EXAMPLES OF PROVING LIMITS OF FUNCTIONS AND OPERATORS.

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1. Limits of linear functions

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 5$.

Claim: $\lim_{x\to 2}(3x+5) = 11$.

Scratch Work: We will need to show that $|(3x+5) - 11| < \epsilon$ provided that $|x-2| < \delta$ where we get to choose δ.

$$|(3x+5) - 11| = |3x-6| < 3|x-2| < 3\delta.$$

In order for $3\delta < \epsilon$, we simply need $\delta \le \epsilon/3$.

Proof: Let $\epsilon > 0$ be given.
Take $\delta = \epsilon/3$.
Let $|x-2| < \delta$.
Then we have

$$\begin{aligned} |(3x+5) - 11| &= |3x-6| = 3|x-2| \\ &< 3\delta = \epsilon \end{aligned}$$

Thus, $\lim_{x\to 2}(3x+5) = 11$ $\square$

Now let's prove something a little bit more difficult.

Claim: $f(x)$ is continuous.

Scratch Work: In order to prove the claim, we need to show that f is continuous at $a \in \mathbb{R}$ for all a. That is, we need to show for arbitrary $a \in \mathbb{R}$ that $\lim_{x\to a} f(x) = f(a)$. In order to do this we will need to show that $|f(x) - f(a)| < \epsilon$ provided that $|x-a| < \delta$.

$$\begin{aligned} |f(x) - f(a)| &= |(3x+5) - (3a+5)| = |3x-3a| = 3|x-a| \\ &< 3\delta \end{aligned}$$

So, in order to make $|f(x) - f(a)| < \epsilon$ provided that $|x-a| < \delta$ if is sufficient to take $\delta = \epsilon/3$. *Notice the similarity to the argument before.*

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Proof: Let $a \in \mathbb{R}$.
Let $\epsilon > 0$ be given.
Take $\delta = \epsilon/3$.
Let $|x - a| < \delta$.
Then we have

$$\begin{aligned} |(3x+5) - (3a+5)| &= |3x - 3a| = 3|x - a| \\ &< 3\delta = \epsilon \end{aligned}$$

Thus, $\lim_{x \to a} f(x) = \lim_{x \to a} (3x + 5) = f(a)$ □

2. Limits of Quadratic functions

Now we will work with a little bit more difficult function. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x^2 + x + 3$.

Claim: $\lim_{x \to 2} g(x) = 13$.

Scratch: For $|x - 2| < \delta$, we will need to show that $|g(x) - 13| < \epsilon$ where ϵ is arbitrary.

$$\begin{aligned} |g(x) - 13| &= |(2x^2 + x + 3) - 13| = |2x^2 + x - 10| \\ &= |(2x + 5)(x - 2)| = |2x + 5||x - 2| \\ &< |2x + 5|\delta = |2(x - 2) + 9|\delta \\ &\leq (2|x - 2| + 9)\,\delta \qquad \text{(by the triangle inequality)} \\ &< (2\delta + 9)\,\delta = 2\delta^2 + 9\delta \\ &< 2\delta + 9\delta \qquad \text{(provided that } \delta < 1\text{)} \\ &= 11\delta. \end{aligned}$$

Thus we must ensure that $\delta < 1$ and that $\delta \leq \epsilon/11$.

Proof: Let $\epsilon > 0$ be given.
Take $\delta = \mathrm{Min}(1/2, \epsilon/11)$.
Suppose that $|x - 2| < \delta$.
Then,

$$\begin{aligned} |g(x) - 13| &= |(2x^2 + x + 3) - 13| = |2x^2 + x - 10| \\ &= |2x + 5||x - 2| \\ &< |2x + 5|\delta = |2(x - 2) + 9|\delta \\ &\leq (2|x - 2| + 9)\,\delta \qquad \text{(by the triangle inequality)} \\ &< (2\delta + 9)\,\delta = 2\delta^2 + 9\delta \\ &< 2\delta + 9\delta \qquad \text{(because } \delta < 1\text{)} \\ &= 11\delta < \epsilon \qquad \text{because } \delta \leq \epsilon/11 \end{aligned}$$

Thus, $\lim_{x \to 2} g(x) = 13$. □

Let us now prove that $g(x)$ is a continuous function. As in the previous section, the argument will resemble the proof that we just completed.

Claim: $g(x)$ is continuous on all of $\mathbb{R}$.

Proof: Suppose that $a \in \mathbb{R}$.
Let $\epsilon > 0$ be given.
Take $\delta = \text{Min}(\frac{1}{2}, \frac{\epsilon}{(|4a+1|+2)})$. Suppose that $|x-a| < \delta$. Then,

$$\begin{aligned}
|g(x) - g(a)| &= |(2x^2 + x + 3) - (2a^2 + a + 3)| = |2(x^2 - a^2) + (x - a)| \\
&= |2(x+a)(x-a) + (x-a)| = |x-a||2(x+a)+1| \\
&< |2(x+a)+1|\delta = |2(x-a+2a)+1|\delta = |2(x-a)+4a+1|\delta \\
&\leq (2|x-a| + |4a+1|)\,\delta \qquad \text{(by the triangle inequality)} \\
&< (2\delta + |4a+1|)\,\delta = 2\delta^2 + |4a+1|\delta \\
&< 2\delta + |4a+1|\delta \qquad \text{(because } \delta < 1\text{)} \\
&= (|4a+1|+2)|\delta < \epsilon \qquad \text{because } \delta \leq \tfrac{\epsilon}{|4a+1|+2}
\end{aligned}$$

Thus, we have shown for all $a \in \mathbb{R}$ that

$$\lim_{x \to a} g(x) = g(a).$$

So, $g(x)$ is continuous on all of $\mathbb{R}$. $\square$

3. Continuity of Operators

We will conclude with an example of a continuous operator on $\mathbb{R}$. Define a binary operation $*$ on $\mathbb{R}$ but $x * y = xy + x + y + 7$.

Claim: The operation $*$ is continuous on $\mathbb{R}$.

Scratch: We will need to show for all $(a,b) \in \mathbb{R}^2$ that $\lim_{(x,y)\to(a,b)} x * y = a * b$. So, given that $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ we will need to show that $|x * y - a * b| < \epsilon$.
We first note that

$$\begin{aligned}
\sqrt{(x-a)^2 + (y-b)^2} < \delta &\Rightarrow (x-a)^2 + (y-b)^2 < \delta^2 \\
&\Rightarrow (x-a)^2 < \delta^2 \quad \text{and} \quad (y-b)^2 < \delta^2 \\
&\Rightarrow |x-a| < \delta \quad \text{and} \quad |y-b| < \delta
\end{aligned}$$

So, we have

$$\begin{aligned}
|x*y-a*b| &= |(xy+x+y+7)-(ab+a+b+7)| = |(xy-ab)+(x-a)+(y-b)| \\
&\leq |xy-ab|+|x-a|+|y-b| \qquad \text{by the triangle inequality} \\
&< |xy-ab|+2\delta \qquad \text{since } |x-a|<\delta \text{ and } |y-b|<\delta \\
&= |(x-a)(y-b)+b(x-a)+a(y-b)|+2\delta \\
&\leq |x-a||y-b|+|b||x-a|+|a||y-b|+2\delta \qquad \text{by the triangle inequality} \\
&< \delta^2+|b|\delta+|a|\delta+2\delta \qquad \text{since } |x-a|<\delta \text{ and } |y-b|<\delta \\
&< \delta+|b|\delta+|a|\delta+2\delta \qquad \text{provided } \delta<1 \\
&= (|a|+|b|+3)\,\delta.
\end{aligned}$$

So, we just need to ensure that $\delta < 1$ and $\delta \leq \frac{\epsilon}{|a|+|b|+3}$.

Proof: Suppose that $(a,b) \in \mathbb{R}^2$.
Let $\epsilon > 0$ be given.
Take $\delta = \text{Min}\left(\frac{1}{2}, \frac{\epsilon}{|a|+|b|+3}\right)$.
Suppose that (x,y) satisfies $\sqrt{(x-a)^2+(y-b)^2} < \delta$.
First, we note that

$$\begin{aligned}
\sqrt{(x-a)^2+(y-b)^2} < \delta &\Rightarrow (x-a)^2+(y-b)^2 < \delta^2 \\
&\Rightarrow (x-a)^2 < \delta^2 \quad \text{and} \quad (y-b)^2 < \delta^2 \\
&\Rightarrow |x-a| < \delta \quad \text{and} \quad |y-b| < \delta
\end{aligned}$$

Thus,

$$\begin{aligned}
|x*y-a*b| &= |(xy+x+y+7)-(ab+a+b+7)| = |(xy-ab)+(x-a)+(y-b)| \\
&\leq |xy-ab|+|x-a|+|y-b| \qquad \text{by the triangle inequality} \\
&< |xy-ab|+2\delta \qquad \text{since } |x-a|<\delta \text{ and } |y-b|<\delta \\
&= |(x-a)(y-b)+b(x-a)+a(y-b)|+2\delta \\
&\leq |x-a||y-b|+|b||x-a|+|a||y-b|+2\delta \qquad \text{by the triangle inequality} \\
&< \delta^2+|b|\delta+|a|\delta+2\delta \qquad \text{since } |x-a|<\delta \text{ and } |y-b|<\delta \\
&< \delta+|b|\delta+|a|\delta+2\delta \qquad \text{since } \delta<1 \\
&= (|a|+|b|+3)\,\delta \\
&\leq \epsilon \qquad \text{because } \delta \leq \tfrac{\epsilon}{|a|+|b|+3}.
\end{aligned}$$

Thus for any $(a,b) \in \mathbb{R}^2$,

$$\lim_{(x,y)\to(a,b)} x*y = a*b,$$

and thus we have shown that $*$ is continuous on all of $\mathbb{R}$ $\square$