MTHSC 3190 Section 5.25

Kevin James

Definition

Let A, B and C be sets. Suppose that

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

Then we define a new function $g \circ f : A \to C$ called the composition of g with f by

$$g \circ f(x) = g(f(x)).$$

Question

Is $g \circ f$ as defined above well defined?

Example

Suppose that

$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ B &= \{5, 6, 7\} \\ C &= \{8, 9, 10, 11\} \\ f &= \{(1, 6), (2, 6), (3, 5), (4, 7)\} \\ g &= \{(5, 8), (6, 10), (7, 11)\} \end{aligned}$$

Then $g \circ f =$ ______________

Note

Note that it is often the case that only one of $f \circ g$ and $g \circ f$ are defined and even if both are defined they are typically different functions.

Example

Let

$$\begin{aligned} A &= \{0,1,2,3,4\} \\ f &= \{(0,1),(1,2),(2,0),(3,0),(4,2)\} \\ g &= \{(0,4),(1,1),(2,3),(3,0),(4,2)\} \end{aligned}$$

Then $f \circ g =$ ________________
$g \circ f =$ ________________

Proposition

Let A, B, C and D be sets and let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then,

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof.

$\square$

Proving function equality

In order to show that two functions f, g are equal, we must prove the following:

1. The functions have the same domain $(\mathrm{dom}(f) = \mathrm{dom}(g))$.
2. For each element of the common domain the two functions take the same value $(\forall x \in \mathrm{dom}(f) = \mathrm{dom}(g),\ f(x) = g(x))$.

Definition (Identity Function)

Let A be a set. The identity function $\mathrm{id}_A : A \to A$ on A is defined by $\mathrm{id}_A(x) = x$.

Proposition

Let A and B be sets. Let $f : A \to B$. Then,

$$f \circ id_A = id_A \circ f = f.$$

Proposition

Let A and B be sets. Suppose that $f : A \to B$ is a bijection. Then,

$$f \circ f^{-1} = id_B \qquad \text{and} \qquad f^{-1} \circ f = id_A.$$