MTHSC 3190 Section 5.25

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Definition

Let A, B and C be sets. Suppose that

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

Then we define a new function $g \circ f : A \to C$ called the composition of g with f by

$$g \circ f(x) = g(f(x)).$$

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Question

Is $g \circ f$ as defined above well defined?

Example

Suppose that

$$
\begin{aligned}
A &= \{1,2,3,4\} \\
B &= \{5,6,7\} \\
C &= \{8,9,10,11\} \\
f &= \{(1,6),(2,6),(3,5),(4,7)\} \\
g &= \{(5,8),(6,10),(7,11)\}
\end{aligned}
$$

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Note

Note that it is often the case that only one of $f \circ g$ and $g \circ f$ are defined and even if both are defined they are typically different functions.

Example

Let

$$\begin{aligned} A &= \{0, 1, 2, 3, 4\} \\ f &= \{(0, 1), (1, 2), (2, 0), (3, 0), (4, 2)\} \\ g &= \{(0, 4), (1, 1), (2, 3), (3, 0), (4, 2)\} \end{aligned}$$

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Proposition

Let A, B, C and D be sets and let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then,

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof.

□

Proving function equality

In order to show that two functions f, g are equal, we must prove the following:

1. The functions have the same domain $(\text{dom}(f) = \text{dom}(g))$.
2. For each element of the common domain the two functions take the same value $(\forall x \in \text{dom}(f) = \text{dom}(g),\ f(x) = g(x))$.

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Proposition

Let A and B be sets. Let $f : A \to B$. Then,

$$f \circ id_A = id_A \circ f = f.$$

Proposition

Let A and B be sets. Suppose that $f : A \to B$ is a bijection. Then,

$$f \circ f^{-1} = id_B \qquad \textit{and} \qquad f^{-1} \circ f = id_A.$$