MTHSC 412 Section 1.2 –Mappings

Kevin James

Definition (Cartesian Product)

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Example

Let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$. Then,

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

Definition (Mapping)

Let A and B be two nonempty sets. A subset f of $A \times B$ is a mapping from A to B provided that for each $a \in A$ there is precisely one $b \in B$ such that $(a, b) \in f$.

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Let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$. Then,

1. $f = \{(1, a), (2, a), (3, b)\}$ is a mapping.
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Notation

If f is a mapping from A to B, then we write

$$f : A \longrightarrow B$$

or

$$A \xrightarrow{f} B.$$

Definition

Suppose that A and B are nonempty sets and that $f \subseteq A \times B$ is a mapping from A to B. If $(a, b) \in f$ we write $f(a) = b$ and say that b is the *image* of a under f.

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$$\begin{aligned} f(1) &= ? \\ f(2) &= ? \\ f(3) &= ? \end{aligned}$$

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Let f be a mapping from A to B. The set A is called the *domain of* f and the set B is called the *codomain of* f. The range (or image) of f is the set

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Example

Suppose that $A = \{1,2,3\}$, $B = \{a,b,c\}$ and $f = \{(1,a),(2,a),(3,b)\}$. Then the range of f is

$$f(A) = \{a,b\}.$$

Definition

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Note

With notation as above we have $f(S) \subseteq B$ and $f^{-1}(T) \subseteq A$.

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Thus for all $y \in \mathbb{Z}$ (the codomain) there is an $x \in \mathbb{Z}$ (the domain) such that $f(x) = y$. $\square$

Definition

A mapping $f : A \to B$ is *one to one* or *injective* if different elements of A get mapped to different elements of B. Equivalently, f is one to one or injective if for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

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Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f = \{(x, x+5) \mid x \in \mathbb{Z}\}$. Then we have already seen that f is a bijection.

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Define $f : \mathbb{Z} \to \mathbb{Z}$ by

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So, f is onto.
(Not One to One): $f(1) = 1 = f(2)$. Thus f is not one to one. □

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There is a solution $x \in \mathbb{Z}$ if and only if b is divisible by 5. Thus f is not onto.
For example there is no $x \in \mathbb{Z}$ such that $f(x) = 6$. $\square$

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Fact

Composition of functions is associative. That is, if $h : A \longrightarrow B$, $g : B \longrightarrow C$ *and* $f : C \longrightarrow D$, *then* $(f \circ g) \circ h = f \circ (g \circ h)$.

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Since the two functions have the same domain and agree on all elements of the domain, they are equal. □

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Corollary

Suppose that $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are both bijections. Then $(f \circ g) : A \longrightarrow C$ is also a bijection.