MTHSC 412 Section 1.2 –Mappings

Kevin James

DEFINITION (CARTESIAN PRODUCT)

For two nonempty sets A and B, the *Cartesian product* of A and B is defined by

$$A \times B = \{(a, b) \quad | \quad a \in A; b \in B\}.$$

EXAMPLE

Let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$. Then,

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

Definition (Mapping)

Let A and B be two nonempty sets. A subset f of $A \times B$ is a mapping from A to B provided that for each $a \in A$ there is precisely one $b \in B$ such that $(a, b) \in f$.

Example

Let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$. Then,

1. $f = \{(1, a), (2, a), (3, b)\}$ is a mapping.
2. $g = \{(1, a), (2, a), (1, b), (3, b)\}$ is not a mapping.

Notation

If f is a mapping from A to B, then we write

$$f : A \longrightarrow B$$

or

$$A \xrightarrow{f} B.$$

DEFINITION

Suppose that A and B are nonempty sets and that $f \subseteq A \times B$ is a mapping from A to B. If $(a, b) \in f$ we write $f(a) = b$ and say that b is the *image* of a under f.

EXAMPLE

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $f = \{(1, a), (2, a), (3, b)\}$. Then

$$\begin{aligned} f(1) &= a \\ f(2) &= a \\ f(3) &= b \end{aligned}$$

Definition

Let f be a mapping from A to B. The set A is called the *domain of f* and the set B is called the *codomain of f*. The range (or image) of f is the set

$$\begin{aligned} f(A) &= \{y \in B \mid y = f(x) \text{ for some } x \in A\} \\ &= \{f(x) \mid x \in A\}. \end{aligned}$$

Example

Suppose that $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $f = \{(1, a), (2, a), (3, b)\}$. Then the range of f is

$$f(A) = \{a, b\}.$$

Definition

Suppose that $f : A \to B$, $S \subseteq A$ and $T \subseteq B$. Then

$$\begin{aligned} f(S) &= \{f(x) \mid x \in S\} \\ &= \{y \in B \mid y = f(x) \text{ for some } x \in S\}. \end{aligned}$$

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

Note

With notation as above we have $f(S) \subseteq B$ and $f^{-1}(T) \subseteq A$.

Example

Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $f = \{(1, a), (2, a), (3, b)\}$. Suppose that $S = \{1, 2\}$ and that $T = \{b, c\}$. Then,

$$\begin{aligned} f(S) &= \{a\} \\ f^{-1}(T) &= \{3\} \end{aligned}$$

Definition

Let $f : A \to B$. f is called *onto* or *surjective* if $f(A) = B$. In this case f is said to be a mapping of A onto B.

Example

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then

- $f = \{(1, a), (2, a), (3, b)\}$ is not onto because $c \notin f(A)$.
- $g = \{(1, a), (2, c), (3, b)\}$ is onto.

Example

Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is given by $f = \{(x, x+5) \mid x \in \mathbb{Z}\}$.
Show that f is onto.

Proof.

Suppose that $y \in \mathbb{Z}$ (the codomain).
Then letting $x = y - 5 \in \mathbb{Z}$ (the domain),
we have

$$f(x) = x + 5 = (y - 5) + 5 = y$$

Thus for all $y \in \mathbb{Z}$ (the codomain) there is an $x \in \mathbb{Z}$ (the domain) such that $f(x) = y$. $\square$

Definition

A mapping $f : A \to B$ is *one to one* or *injective* if different elements of A get mapped to different elements of B. Equivalently, f is one to one or injective if for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

Example

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then

- $f = \{(1, a), (2, a), (3, b)\}$ is not one to one because $f(1) = f(2)$.
- $g = \{(1, a), (2, c), (3, b)\}$ is one to one.

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f = \{(x, x+5) \quad | \quad x \in \mathbb{Z}\}$. Show that f is one to one.

Proof.

Suppose that $a, b \in \mathbb{Z}$ and that $f(a) = f(b)$.
Then

$$\begin{aligned} & f(a) = f(b) \\ \Rightarrow \quad & a + 5 = b + 5 \\ \Rightarrow \quad & a = b \end{aligned}$$

Thus if $a \neq b$ then $f(a) \neq f(b)$. So, f is injective. $\square$

One to One Correspondence, Bijection

Definition

A mapping $f : A \to B$ is a *one to one correspondence* or a *bijection* if f is both injective and surjective.

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f = \{(x, x+5) \quad | \quad x \in \mathbb{Z}\}$. Then we have already seen that f is a bijection.

Example

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Show that f is onto but not one to one.

Proof.

(Onto): Suppose that $b \in \mathbb{Z}$ (the codomain).
We note that selecting $x = 2b$ and $y = 2b - 1$ from the domain $\mathbb{Z}$ yeilds $f(x) = f(y) = b$.
Thus for any $b \in \mathbb{Z}$ there is an $x \in \mathbb{Z}$ such that $f(x) = b$.
So, f is onto.
(Not One to One): $f(1) = 1 = f(2)$. Thus f is not one to one. $\square$

Example

Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 5x$. Show that f is one to one but not onto.

Proof.

(Injective): Suppose that $a, b \in \mathbb{Z}$ and $f(a) = f(b)$
$\Rightarrow 5a = 5b \Rightarrow a = b$.
Thus if $a \neq b$ then $f(a) \neq f(b)$ and f is injective.
(Not onto): Let $b \in \mathbb{Z}$
Then $f(x) = b \Rightarrow 5x = b$
There is a solution $x \in \mathbb{Z}$ if and only if b is divisible by 5. Thus f is not onto.
For example there is no $x \in \mathbb{Z}$ such that $f(x) = 6$. □

Definition

Let $g : A \to B$ and $f : B \to C$. Then the *composite mapping* $f \circ g : A \to C$ is defined by

$$f \circ g(x) = f(g(x)).$$

Example

Let $A = \{x \in \mathbb{Z} \quad | \quad x \geq 0\}$ and let $B = \{x \in \mathbb{Z} \quad | \quad x \leq 0\}$.
Suppose that $f : \mathbb{Z} \to A$ and $g : A \to B$ are defined by

$$f(x) = x^4 \qquad \text{and} g(x) = -x - 3.$$

Then

$$g \circ f(x) = g(f(x)) = g(x^4) = -x^4 - 3.$$

Fact

Composition of functions is associative. That is, if $h : A \longrightarrow B$, $g : B \longrightarrow C$ and $f : C \longrightarrow D$, then $(f \circ g) \circ h = f \circ (g \circ h)$.

Proof.

Note that $(f \circ g) : B \longrightarrow D$. Thus $((f \circ g) \circ h) : A \longrightarrow D$.
Similarly, $(g \circ h) : A \longrightarrow C$. Thus $(f \circ (g \circ h)) : A \longrightarrow D$.
So the two functions have the same domain.
Also for any $x \in A$, we have

$$\begin{aligned} ((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x))) \\ &= f((g \circ h)(x)) \\ &= (f \circ (g \circ h))(x). \end{aligned}$$

Since the two functions have the same domain and agree on all elements of the domain, they are equal. $\square$

Theorem

Suppose that $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are both surjective. Then $(f \circ g) : A \longrightarrow C$ is also surjective.

Theorem

Suppose that $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are both injective. Then $(f \circ g) : A \longrightarrow C$ is also injective.

Corollary

Suppose that $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are both bijections. Then $(f \circ g) : A \longrightarrow C$ is also a bijection.