MTHSC 412 Section 1.5 –Permutations and Inverses

Kevin James

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- We will denote by $\mathcal{M}(A)$ the set of all mappings form A to A.
- In the special case that the set $A = \{1, 2, \ldots, n\}$, we use the notation $S_n = \mathcal{S}(A)$.

Remark

Suppose that A is a nonempty set. Then composition of functions is an associative binary operation on $\mathcal{M}(A)$. The identity element I_A of $\mathcal{M}(A)$ under composition of functions is given by

$$I_A(x) = x \quad \text{for all } x \in A.$$

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Thus I_A is the identity element. $\square$

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However, $f \circ g(x) = f(g(x)) = \begin{cases} f(x/2) \\ f(4) \end{cases} = \begin{cases} x & \text{if } x \text{ is even,} \\ 8 & \text{if } x \text{ is odd.} \end{cases}$

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So, $f \circ g \neq I_{\mathbb{Z}}$ and g is not a right inverse of f.

LEMMA

Let A be a nonempty set and let $f : A \longrightarrow A$. Then f is injective if and only if f has a left inverse.

Proof.

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Define $g \in \mathcal{M}(A)$ as follows.

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$$g(x) = \begin{cases} y & \text{if there exists } y \in A \text{ such that } f(y) = x, \\ a_0 & \text{otherwise} \end{cases}$$

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So, g is a *well-defined* mapping.

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So, g is a *well-defined* mapping.
For all $x \in A$ we have $g \circ f(x) = g(f(x)) = x$.
So, $g \circ f = I_A$ and g is a left inverse of f. $\square$

Lemma

Let A be a nonempty set and $f : A \longrightarrow A$. Then f is surjective if and only if f has a right inverse.

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Thus f is invertible. $\square$

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