# MTHSC 412 Section 1.6 – Matrices

Kevin James

## Definition

1. Let $S$ be a set. An $m \times n$ matrix over $S$ will be an array

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

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3. We denote the set of $m \times n$ matrices over $S$ by $M_{m \times n}(S)$.
4. We denote the set of $n \times n$ (square) matrices over $S$ by $M_n(S)$.

**Definition**

Suppose that $A, B \in M_{m \times n}(\mathbb{R})$. Then we define their sum to be the $m \times n$ matrix $C$ with

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1. *Addition is a binary operation on* $M_{m \times n}(\mathbb{R})$.
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3. $M_{m \times n}(\mathbb{R})$ *contains an identity element with respect to addition, namely the matrix with all zero entries.*

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5. *Addition is commutative in $M_{m \times n}(\mathbb{R})$.*

**Definition**

Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$. Then we define the product $AB$ to be the $m \times p$ matrix $C = AB$ with

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

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That is,

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That is,

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Or put another way, matrix multiplication is defined so that

$$A(B\vec{v}) = (AB)\vec{v}.$$

### NOTE (MATRIX MULTIPLICATION IS NOT COMMUTATIVE)

Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{p\times r}(\mathbb{R})$.

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Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{p\times r}(\mathbb{R})$.

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3. The product $BA$ is defined only when $m = r$.

# Properties of Matrix Multiplication

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5. So, $AB$ and $BA$ are both defined only when $m = r$ and $n = p$.

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5. So, $AB$ and $BA$ are both defined only when $m = r$ and $n = p$.
6. If $m = r$ and $n = p$ then $AB$ is $m \times m$ and $BA$ is $n \times n$.

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5. So, $AB$ and $BA$ are both defined only when $m = r$ and $n = p$.
6. If $m = r$ and $n = p$ then $AB$ is $m \times m$ and $BA$ is $n \times n$.
7. So, $AB$ and $BA$ are both defined and have the same dimensions only if $m = n = p = r$.

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5. So, $AB$ and $BA$ are both defined only when $m = r$ and $n = p$.
6. If $m = r$ and $n = p$ then $AB$ is $m \times m$ and $BA$ is $n \times n$.
7. So, $AB$ and $BA$ are both defined and have the same dimensions only if $m = n = p = r$.
8. Even when $m = n = p = r$, it still may be the case that $AB \neq BA$. In fact this is the usual case.

**Theorem**

*Matrix multiplication is associative. That is if* $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$ *and* $C \in M_{p\times r}(\mathbb{R})$, *then* $(AB)C = A(BC)$.

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**Note**

This follows from our recognition of matrix multiplication as composition of functions and our proof that compositions of functions is associative.

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&= \sum_{s=1}^{p} c_{s,j} \sum_{k=1}^{n} a_{i,k}b_{k,s} = \sum_{s=1}^{p} c_{s,j}[AB]_{i,s}
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First we note that $AB$ is $m \times p$ and thus $(AB)C$ is $m \times r$.
Also, $BC$ is $n \times r$ and thus $A(BC)$ is $m \times r$.
Thus $A(BC)$ and $(AB)C$ both have the same dimensions.
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[A(BC)]_{i,j} &= \sum_{k=1}^{n} a_{i,k}[BC]_{k,j} = \sum_{k=1}^{n} a_{i,k} \sum_{s=1}^{p} b_{k,s}c_{s,j} \\
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Thus $A(BC) = (AB)C$. $\square$

**Theorem**

*Let $A \in M_{m\times n}(\mathbb{R})$, $B, C \in M_{n\times p}(\mathbb{R})$ and $D \in M_{p\times r}(\mathbb{R})$. Then,*

1. $A(B + C) = AB + AC$.
2. $(B + C)D = BC + CD$.

## Definition

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### Theorem

*Suppose that $A \in M_{m \times n}(\mathbb{R})$. Then,*

1. $I_m A = A$.
2. $A I_n = A$.

### Definition

Let $*$ be a binary operation on a nonempty set $A$.

1. If $e \in A$ satisfies $e * a = a$ for all $a \in A$, then $e$ is said to be a *left identity* for $A$ with respect to $*$.

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3. Note that if $e \in A$ is both a left and right identity then it is an identity.

## Note

1. We call $I_m$ a left identity for $M_{m\times n}(\mathbb{R})$ with respect to matrix multiplication, even though matrix multiplication is not a binary operation on $M_{m\times n}(\mathbb{R})$ unless $m = n$.

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3. $I_n$ is an identity for $M_n(\mathbb{R})$ with respect to matrix multiplication.

### Theorem

1. *Addition is an associative binary operation on* $M_n(\mathbb{R})$.
2. *The zero matrix is an identity for* $M_n(\mathbb{R})$ *w. r. t. addition .*
3. *Each matrix* $A \in M_n(\mathbb{R})$ *has an inverse w. r. t. addition , namely* $-A$.
4. *Addition is commutative.*
5. *Multiplication is an associative binary operation on* $M_n(\mathbb{R})$.
6. $I_n$ *is an identity for* $M_n(\mathbb{R})$ *w. r. t. matrix multiplication.*
7. $A \in M_n(\mathbb{R})$ *has an inverse w. r. t. matrix multiplication if and only if* $\det(A) \neq 0$ *(from linear algebra).*
8. *For* $A, B, C \in M_n(\mathbb{R})$, *we have* $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$.