MTHSC 412 Section 1.6 – Matrices

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Definition

1. Let S be a set. An $m \times n$ matrix over S will be an array

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

   with m rows and n columns and elements $a_{i,j} \in S$.
2. Two matrices A and B over S are equal if they have the same dimensions and if $a_{i,j} = b_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.
3. We denote the set of $m \times n$ matrices over S by $M_{m \times n}(S)$.
4. We denote the set of $n \times n$ (square) matrices over S by $M_n(S)$.

Addition of Matrices in $M(\mathbb{R})$

Definition

Suppose that $A, B \in M_{m\times n}(\mathbb{R})$. Then we define their sum to be the $m \times n$ matrix C with

$$c_{i,j} = a_{i,j} + b_{i,j}$$

Theorem (Properties of Addition on $M_{m\times n}(\mathbb{R})$)

1. *Addition is a binary operation on $M_{m\times n}(\mathbb{R})$.*
2. *Addition is associative on $M_{m\times n}(\mathbb{R})$.*
3. *$M_{m\times n}(\mathbb{R})$ contains an identity element with respect to addition, namely the matrix with all zero entries.*
4. *Each element of $M_{m\times n}(\mathbb{R})$ has an additive inverse in $M_{m\times n}(\mathbb{R})$.*
5. *Addition is commutative in $M_{m\times n}(\mathbb{R})$.*

DEFINITION

Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$. Then we define the product AB to be the $m \times p$ matrix $C = AB$ with

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Note

Matrix multiplication encodes composition of functions.
Let $A \in M_{m \times n}(\mathbb{R})$ and $B \in M_{n \times p}(\mathbb{R})$.
Define functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ by

$$f(\vec{v}) = A\vec{v} \quad \text{and} \quad g(\vec{w}) = B\vec{w}.$$

Then $f \circ g : \mathbb{R}^p \to \mathbb{R}^m$ is a linear map with matrix $C = AB$
That is,

$$f \circ g(\vec{v}) = C\vec{v} \quad \text{where } C = AB.$$

Or put another way, matrix multiplication is defined so that

$$A(B\vec{v}) = (AB)\vec{v}.$$

Properties of Matrix Multiplication

Note (Matrix Multiplication is not Commutative)

Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{p\times r}(\mathbb{R})$.

1. The product AB is defined only when $n = p$.
2. If $n = p$ then AB is an $m \times r$ matrix.
3. The product BA is defined only when $m = r$.
4. If $m = r$ then BA is a $p \times n$ matrix.
5. So, AB and BA are both defined only when $m = r$ and $n = p$.
6. If $m = r$ and $n = p$ then AB is $m \times m$ and BA is $n \times n$.
7. So, AB and BA are both defined and have the same dimensions only if $m = n = p = r$.
8. Even when $m = n = p = r$, it still may be the case that $AB \neq BA$. In fact this is the usual case.

Theorem

Matrix multiplication is associative. That is if $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$ *and* $C \in M_{p\times r}(\mathbb{R})$, *then* $(AB)C = A(BC)$.

Note

This follows from our recognition of matrix multiplication as composition of functions and our proof that compositions of functions is associative.

Proof.

First we note that AB is $m \times p$ and thus $(AB)C$ is $m \times r$.
Also, BC is $n \times r$ and thus $A(BC)$ is $m \times r$.
Thus $A(BC)$ and $(AB)C$ both have the same dimensions.
Also, for $1 \le i \le m$ and $1 \le j \le r$, we have

$$\begin{aligned}
[A(BC)]_{i,j} &= \sum_{k=1}^{n} a_{i,k}[BC]_{k,j} = \sum_{k=1}^{n} a_{i,k} \sum_{s=1}^{p} b_{k,s}c_{s,j} \\
&= \sum_{s=1}^{p}\sum_{k=1}^{n} a_{i,k}(b_{k,s}c_{s,j}) = \sum_{s=1}^{p}\sum_{k=1}^{n}(a_{i,k}b_{k,s})c_{s,j} \\
&= \sum_{s=1}^{p} c_{s,j} \sum_{k=1}^{n} a_{i,k}b_{k,s} = \sum_{s=1}^{p} c_{s,j}[AB]_{i,s} \\
&= \sum_{s=1}^{p} [AB]_{i,s}c_{s,j} = [(AB)C]_{i,j}
\end{aligned}$$

Thus $A(BC) = (AB)C$. $\square$

THEOREM

Let $A \in M_{m\times n}(\mathbb{R})$, $B, C \in M_{n\times p}(\mathbb{R})$ *and* $D \in M_{p\times r}(\mathbb{R})$. *Then,*

1. $A(B + C) = AB + AC$.
2. $(B + C)D = BC + CD$.

Definition

Define $I_n \in M_n(\mathbb{R})$ by

$$[I_n]_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Suppose that $A \in M_{m \times n}(\mathbb{R})$. Then,

1. $I_m A = A$.
2. $A I_n = A$.

Definition

Let $*$ be a binary operation on a nonempty set A.

1. If $e \in A$ satisfies $e * a = a$ for all $a \in A$, then e is said to be a *left identity* for A with respect to $*$.
2. If $e \in A$ satisfies $a * e = a$ for all $a \in A$, then e is said to be a *right identity* for A with respect to $*$.
3. Note that if $e \in A$ is both a left and right identity then it is an identity.

Note

1. We call I_m a left identity for $M_{m\times n}(\mathbb{R})$ with respect to matrix multiplication, even though matrix multiplication is not a binary operation on $M_{m\times n}(\mathbb{R})$ unless $m = n$.
2. Similarly we call I_n a right identity for $M_{m\times n}(\mathbb{R})$ with respect to matrix multiplication.
3. I_n is an identity for $M_n(\mathbb{R})$ with respect to matrix multiplication.

THEOREM

1. *Addition is an associative binary operation on* $M_n(\mathbb{R})$.
2. *The zero matrix is an identity for* $M_n(\mathbb{R})$ *w. r. t. addition .*
3. *Each matrix* $A \in M_n(\mathbb{R})$ *has an inverse w. r. t. addition , namely* $-A$.
4. *Addition is commutative.*
5. *Multiplication is an associative binary operation on* $M_n(\mathbb{R})$.
6. I_n *is an identity for* $M_n(\mathbb{R})$ *w. r. t. matrix multiplication.*
7. $A \in M_n(\mathbb{R})$ *has an inverse w. r. t. matrix multiplication if and only if* $\det(A) \neq 0$ *(from linear algebra).*
8. *For* $A, B, C \in M_n(\mathbb{R})$, *we have* $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$.