MTHSC 412 Section 1.7 – Relations

Kevin James

DEFINITION

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Suppose that $A = \{a, b, c\}$ and $R = \{(a, b), (b, c), (c, a)\}$. Then we have aRb and $a\not{R}c$.

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Some well known relations on the integers are $<, >, \leq, \geq$ and $=$. Also, we have seen the $\subseteq$ relation on sets whose elements are sets.

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A relation R on a nonempty set A is an *equivalence relation* if the following conditions hold for $x, y, z \in A$.

1. xRx for all $x \in A$. **(Reflexive Property)**
2. If xRy then yRx also. **(Symmetric Property)**
3. If xRy and yRz then xRz also. **(Transitive Property)**

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We will consider the relation on $\mathbb{Z}$ defined as the set $\{(x, y) \in \mathbb{Z}^2 \quad | \quad (x - y) \text{ is divisible by } 4\}$. If (a, b) is in this set, we write $a \equiv b \pmod 4$.

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Fact

$\equiv \pmod 4$ is an equivalence relation on $\mathbb{Z}$.

Proof.

Reflexive: Suppose that $x \in \mathbb{Z}$. Then $x - x = 0$ which is divisible by 4. So, $\equiv \pmod 4$ is reflexive.

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So, $x - z = (x - y) + (y - z) = 4k + 4m = 4(k + m)$.

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Since $\equiv \pmod 4$ is reflexive, symmetric and transitive, it is an equivalence relation. □

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Let $\{A_i\}_{i \in I}$ be a collection of subsets of a nonempty set A. We say that $\{A_i\}_{i \in I}$ is a *partition* of A if the following conditions are satisfied.

1. $A_i \neq \emptyset$ for all $i \in I$.
2. $A = \cup_{i \in I} A_i$.
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Equivalence Classes Form a Partition

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Fact

- *If R is an equivalence relation on a nonempty set A then $\{[a] \quad | \quad a \in A\}$ is a partition of A.*
- *If $P = \{A_i\}_{i \in I}$ is a partition of A then there is an equivalence relation R on A such that the equivalence classes of R are precisely the parts A_i of P. To see this just define R by aRb if and only if a and b are in the same part of P.*