MTHSC 412 Section 2.2 – Mathematical Induction

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Principle of Mathematical Induction

Suppose that $P(n)$ is a statement about the integer n and that the following two conditions are satisfied.

1. $P(a)$ is true for some integer a.
2. If $P(k)$ is true for some $k \geq a$ then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq a$.

Proof by Induction

To prove that a statement $P(n)$ is true for all $n \geq a$, where $a \in \mathbb{Z}$.

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2. **Induction Hypothesis:** Assume that $P(k)$ is true for some $k \geq a$.
3. **Induction Step:** Under the above assumption, prove that $P(k+1)$ is true.
4. Deduce that $P(n)$ is true for all $n \geq a$ by induction.

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Thus by induction we see that $\sum_{i=1}^{k}(2i-1) = k^2$ for all $k \geq 1$. $\square$

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Based on this data we might conjecture that ....

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$$5(k+1) + 1 = (5k + 1) + 5.$$

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It follows by induction that $n^2 > 5n + 1$ for all $n \geq 6$. □

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Proof by Strong Induction

To prove that $P(n)$ is true for all $n \geq a$:

1. **Basis Step:** Check that $P(a)$ is true.
2. **Induction Hypothesis:** Assume that $P(n)$ is true for all $a \leq n \leq k$ for some $k \geq a$.
3. **Induction Step:** Under the above assumption, prove that $P(k+1)$ is true.
4. Deduce that $P(n)$ is true for all $n \geq a$ by strong induction.

Definition

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Theorem

Every integer $n \geq 2$ has a prime divisor.

PROOF.

Basis Step: 2 is prime and divides itself. So, the statement is true for $n = 2$.

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