# MTHSC 412 Section 2.2 – Mathematical Induction

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**Principle of Mathematical Induction**

Suppose that $P(n)$ is a statement about the integer $n$ and that the following two conditions are satisfied.

1. $P(a)$ is true for some integer $a$.
2. If $P(k)$ is true for some $k \geq a$ then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq a$.

### Proof by Induction

To prove that a statement $P(n)$ is true for all $n \geq a$, where $a \in \mathbb{Z}$.

1. **Basis Step:** Check that $P(a)$ is true.
2. **Induction Hypothesis:** Assume that $P(k)$ is true for some $k \geq a$.
3. **Induction Step:** Under the above assumption, prove that $P(k+1)$ is true.
4. Deduce that $P(n)$ is true for all $n \geq a$ by induction.

**Fact**

*For any* $n \geq 1$, $\sum_{i=1}^{n}(2i-1) = n^2$.

**PROOF.**

**Basis Step:** When $n = 1$ we have $\sum_{i=1}^{1}(2i-1) = 1 = 1^2$. So the statement is true when $n = 1$.

**Induction Hypothesis:** Suppose that $\sum_{i=1}^{k}(2i-1) = k^2$ for some $k \geq 1$.

**Induction Step:** Then,

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1) &= \left(\sum_{i=1}^{k}(2i-1)\right) + 2(k+1) - 1 \\ &= k^2 + 2k + 1, \quad \text{by our induction hypothesis.} \\ &= (k+1)^2. \end{aligned}$$

Thus by induction we see that $\sum_{i=1}^{k}(2i-1) = k^2$ for all $k \geq 1$. $\square$

Consider the following data.

| $n$ | $5n+1$ | $n^2$ |
|---|---|---|
| 1 | 6 | 1 |
| 2 | 11 | 4 |
| 3 | 16 | 9 |
| 4 | 21 | 16 |
| 5 | 26 | 25 |
| 6 | 31 | 36 |
| 7 | 36 | 49 |
| 8 | 41 | 64 |

Based on this data we might conjecture that ....

## Fact

*For $n \geq 6$, $n^2 > 5n + 1$.*

## Proof.

**Basis Step:** From our table, we see that $n^2 > 5n + 1$ for $n = 6, 7$ and 8.

**Induction Hypothesis:** Assume that $k^2 > 5k + 1$ for some $k \geq 6$.

**Induction Step:** For $n = k + 1$, we have

$$\begin{aligned} 5(k+1)+1 &= (5k+1)+5. \\ &< k^2+5, \quad \text{by our induction hypothesis.} \\ &< k^2+2k+1 \quad \text{since } k \geq 6. \\ &= (k+1)^2. \end{aligned}$$

It follows by induction that $n^2 > 5n + 1$ for all $n \geq 6$. □

An proof technique which is equivalent to induction but more convenient to use in many cases is *strong induction*.

**Proof by Strong Induction**

To prove that $P(n)$ is true for all $n \geq a$:

1. **Basis Step:** Check that $P(a)$ is true.
2. **Induction Hypothesis:** Assume that $P(n)$ is true for all $a \leq n \leq k$ for some $k \geq a$.
3. **Induction Step:** Under the above assumption, prove that $P(k+1)$ is true.
4. Deduce that $P(n)$ is true for all $n \geq a$ by strong induction.

# Example

### Definition

An integer $p$ is prime is $p > 1$ and if the only integers which divide $p$ evenly are $\pm 1$ and $\pm p$.

### Theorem

*Every integer $n \geq 2$ has a prime divisor.*

**Proof.**

**Basis Step:** 2 is prime and divides itself. So, the statement is true for $n = 2$.

**Induction Hypothesis:** Suppose that each $2 \leq n \leq k$ has a prime divisor for some $k \geq 2$.

**Induction Step:** Now let us consider $n = k + 1$.

We will consider two cases. Either $k + 1$ is prime or it is not.

**Case 1:** If $k + 1$ is prime then $k + 1$ is a prime divisor of itself.

**Case 2:** If $k + 1$ is not prime then we can write

$$k + 1 = \ell m, \quad \text{for some } 1 < \ell \leq m < k + 1.$$

Since $2 \leq m \leq k$, our induction hypothesis implies that $m$ is divisible by some prime $p$.

That is $m = pr$ for some $r \in \mathbb{Z}$.

So, we have $k + 1 = \ell m = \ell rp$, and $p$ divides $k + 1$.

Thus in either case, we have shown that $k + 1$ has a prime divisor and our theorem follows by strong induction. $\square$