MTHSC 412 Section 2.5 –Congruence of Integers

Kevin James

Congruence modulo n is an equivalence relation on $\mathbb{Z}$

Definition

Let $n > 1$. be an integer. For $x, y \in \mathbb{Z}$, we say that x is congruent to y modulo n and write $x \equiv y \pmod{n}$ if $n|(x - y)$.

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Theorem

If $n > 1$ is an integer then $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$.

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Then $n|(x - y) \Rightarrow (x - y) = nk$ for some $k \in \mathbb{Z}$.

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So, $(y - x) = n(-k)$. Thus, $n|(y - x)$ and $y \equiv x \pmod{n}$.

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Then $(x - y) = nk$ and $(y - z) = nm$ for some $k, m \in \mathbb{Z}$.
So, $(x - z) = (x - y) + (y - z) = n(k + m)$ and $n|(x - z)$.

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So, $(x - z) = (x - y) + (y - z) = n(k + m)$ and $n|(x - z)$.
Thus $x \equiv z \pmod{n}$ and $\equiv \pmod{n}$ is transitive.
Since $\equiv \pmod{n}$ is reflexive, symmetric and transitive, it is an equivalence relation on $\mathbb{Z}$. □

Fact

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$$\begin{aligned} x &= nq_1 + r_1 \\ y &= nq_2 + r_2 \end{aligned}$$

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Thus $x - y = n(q_1 - q_2) + (r_1 - r_2)$ with $-n < (r_1 - r_2) < n$. Now note that $n | (x - y)$ if and only if $n | (r_1 - r_2)$.

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Thus $x - y = n(q_1 - q_2) + (r_1 - r_2)$ with $-n < (r_1 - r_2) < n$.
Now note that $n|(x - y)$ if and only if $n|(r_1 - r_2)$.
Finally since $-n < (r_1 - r_2) < n$, $n|(r_1 - r_2)$ if and only if $r_1 = r_2$. □

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Since, $x - r = nq$, $x \equiv r \pmod{n}$.
Thus each integer is in one of the congruence classes: $[0], [1], \ldots [n-1]$.
The fact that these are distinct follows from our last fact. □

Theorem

If $a \equiv b \pmod{n}$ *and* $x \in \mathbb{Z}$ *then*

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Addition and Multiplication Properties

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Proof.

Suppose that $a \equiv b \pmod{n}$. Then $(a - b) = nk$ for some $k \in \mathbb{Z}$. Thus $(a + x) - (b + x) = a - b = nk$ and $ax - bx = x(a - b) = xnk$ and the result follows. □

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Suppose that $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$. *Then*

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The proof of the other congruence is similar and is left as an exercise. □

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□

Theorem

If $(a, n) = 1$, the congruence $ax \equiv b \pmod{n}$ has a solution $x \in \mathbb{Z}$ and the solution is unique modulo *n, which means that any two such solutions are congruent modulo n.*

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Uniqueness modulo n: Suppose that $x, y \in \mathbb{Z}$ are both solutions.
Then $ax \equiv b \equiv ay \pmod{n}$ and $(a, n) = 1$.
By the cancellation law, it follows that $x \equiv y \pmod{n}$. □

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which implies that $b|k$.

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Thus, $k = br$ for some $r \in \mathbb{Z}$.

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Then $c = ak = abr$.
Thus $(ab)|c$. □

Corollary

If $n_1, n_2, \ldots, n_k$ is a set of pairwise coprime integers and if $n_i|c$ for $1 \leq i \leq k$, then $(n_1 n_2 \ldots n_k)|c$.

Theorem

Let $n_1, n_2, \ldots, n_k$ be pairwise coprime integers. Let $a_1, \ldots, a_k \in \mathbb{Z}$. There is $x \in \mathbb{Z}$ satisfying the system of congruences

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \vdots \\ x \equiv a_k \pmod{n_k}. \end{cases}$$

Furthermore, the solution is unique modulo $(n_1 n_2 \ldots n_k)$.

Proof.

Existence: Let $N = (n_1 n_2 \dots n_k)$ and let $N_i = \frac{N}{n_i} = n_1 n_2 \dots n_{i-1} n_{i+1} \dots n_k$.

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$$\text{Then } N_i M_i \equiv \begin{cases} 0 \pmod{n_j} & \text{if } i \neq j, \\ 1 \pmod{n_i} & \end{cases}.$$

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Now let $x = \sum_{i=1}^{k} a_i N_i M_i$.

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Proof.

Existence: Let $N = (n_1 n_2 \dots n_k)$ and let
$N_i = \frac{N}{n_i} = n_1 n_2 \dots n_{i-1} n_{i+1} \dots n_k$.
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PROOF.

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$(n_1 n_2 \ldots n_k) | (x - y)$.
Thus $x \equiv y \pmod{N}$ as desired. □