MTHSC 412 Section 2.6 –Congruence Classes

Kevin James

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Note

It is also common to omit the brackets and simply write

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}.$$

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Under the equivalence relation $\equiv \pmod{n}$ *on* $\mathbb{Z}$, $[a] = [b]$ *if and only if* $a \equiv b \pmod{n}$.

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Since, we also have $a \equiv b \pmod{n}$, it follows from transitivity that $x \equiv b \pmod{n}$.

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Thus $x \in [b]$.
So, $[a] \subseteq [b]$.
Similarly, we can show that $[b] \subseteq [a]$ and thus $[a] = [b]$. □

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Recall that $[1] = [5]$ and $[2] = [6]$.

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We must make sure that this is always the case for addition to be *well defined*.

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2. *Addition on* $\mathbb{Z}_n$ *is associative.*
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4. $[0]$ *is the additive identity for* $\mathbb{Z}_n$.
5. *Each* $a \in \mathbb{Z}_n$ *has an additive inverse,* $[-a]$ *in* $\mathbb{Z}_n$.

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Thus $a + b \equiv c + d \pmod{n}$ from results of section 2.4.

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Then $[0] + [a] = [a] + [0]$

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(4.) Suppose that $[a] \in \mathbb{Z}_n$.
Then $[0] + [a] = [a] + [0] = [a + 0] = [a]$.
(5.) Note that $[-a] = [n - a] \in \mathbb{Z}_n$

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(5.) Note that $[-a] = [n - a] \in \mathbb{Z}_n$ and
$[a] + [-a] = [a + (-a)] = [0]$. $\square$

Definition (Multiplication in $\mathbb{Z}_n$)

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Theorem

1. *Multiplication is a well defined binary operation on* $\mathbb{Z}_n$.
2. *Multiplication on* $\mathbb{Z}_n$ *is associative.*
3. *Multiplication on* $\mathbb{Z}_n$ *is commutative.*
4. $[1]$ *is the multiplicative identity for* $\mathbb{Z}_n$.

Proof.

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3. $[a][b] = [ab] = [ba] = [b][a]$.

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3. $[a][b] = [ab] = [ba] = [b][a]$.
4. Let $[a] \in \mathbb{Z}_n$. Then, $[a][1] = [1][a] = [(1)(a)] = [a]$.

$\square$

Example

Consider the multiplication table for $\mathbb{Z}_6$.

$\times$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

Definition

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Example

From the multiplication table for $\mathbb{Z}_6$, we see that $[2], [3]$ and $[4]$ are zero divisors in $\mathbb{Z}_6$.

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Thus, $[a][s] = [as] = [1]$.

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