Consider the Cayley tables for \( G = \{ \pm 1, \pm i \} = \langle i \rangle \) and \( \mathbb{Z}_4 \).
**Example**

Consider the Cayley tables for $G = \{ \pm 1, \pm i \} = \langle i \rangle$ and $\mathbb{Z}_4$.

<table>
<thead>
<tr>
<th>.</th>
<th>1</th>
<th>$i$</th>
<th>$i^2 = -1$</th>
<th>$i^3 = -i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>$i^3 = -i$</td>
</tr>
<tr>
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<td>$i^3 = -i$</td>
<td>1</td>
</tr>
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<td>1</td>
<td>$i$</td>
</tr>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
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<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Consider the Cayley tables for $G = \{ \pm 1, \pm i \} = \langle i \rangle$ and $\mathbb{Z}_4$.

\[
\begin{array}{c|ccccc}
. & 1 & i & i^2 = -1 & i^3 = -i \\
\hline
1 & 1 & i & i^2 = -1 & i^3 = -i \\
i & i & i^2 = -1 & i^3 = -i & 1 \\
i^2 = -1 & i^2 = -1 & i^3 = -i & 1 & i \\
i^3 = -i & i^3 = -i & 1 & i & i^2 = -1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\]

Do you notice similarities? Are these the same in some sense?
• There are 4 reflections (2 through the diagonals and 2 through lines bisecting opposite sides).

• There are 4 rotations through $0, \pi/2, \pi$ and $3\pi/2$ radians.

• These functions form a group under composition of functions.

• The Cayley table for this group which is denoted $D_4$ is

\[
\begin{array}{cccc}
e & r & r^2 & r^3 \\
e & e & r & r^2 \\
r & r & r^2 & r^3 \\
r^2 & r^2 & r^3 & e \\
r^3 & r^3 & e & r \\
v & h & d_1 & d_2 \\
v & v & h & d_1 \\
h & h & v & d_1 \\
d_1 & d_1 & h & v \\
d_2 & d_2 & v & d_1 \\
d_2 & d_2 & v & d_1 \\
\end{array}
\]
Rigid Motions of a Square

- There are 4 reflections (2 through the diagonals and 2 through lines bisecting opposite sides). We will denote these by $d_1, d_2, v$ and $h$. 

\[
\begin{array}{|c|c|c|c|}
\hline
& e & r & r^2 \\
\hline 
\hline
e & e & r & r^2 \\
\hline
r & r & e & r^2 \\
\hline
r^2 & r^2 & r & e \\
\hline
\end{array}
\]
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<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$r$</th>
<th>$r^2$</th>
<th>$r^3$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$h$</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$r$</td>
<td>$r^2$</td>
<td>$r^3$</td>
<td>$d_1$</td>
<td>$d_2$</td>
<td>$h$</td>
<td>$v$</td>
</tr>
<tr>
<td>$r$</td>
<td>$r$</td>
<td>$r^2$</td>
<td>$r^3$</td>
<td>$e$</td>
<td>$v$</td>
<td>$h$</td>
<td>$d_1$</td>
<td>$d_2$</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$r^2$</td>
<td>$r^3$</td>
<td>$e$</td>
<td>$r$</td>
<td>$d_2$</td>
<td>$d_1$</td>
<td>$v$</td>
<td>$h$</td>
</tr>
<tr>
<td>$r^3$</td>
<td>$r^3$</td>
<td>$e$</td>
<td>$r$</td>
<td>$r^2$</td>
<td>$h$</td>
<td>$v$</td>
<td>$d_2$</td>
<td>$d_1$</td>
</tr>
<tr>
<td>$d_1$</td>
<td>$d_1$</td>
<td>$h$</td>
<td>$d_2$</td>
<td>$v$</td>
<td>$e$</td>
<td>$r^2$</td>
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<td>$d_2$</td>
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<td>$h$</td>
<td>$r^2$</td>
<td>$e$</td>
<td>$r^3$</td>
<td>$r$</td>
</tr>
<tr>
<td>$h$</td>
<td>$h$</td>
<td>$d_2$</td>
<td>$v$</td>
<td>$d_1$</td>
<td>$r^3$</td>
<td>$r$</td>
<td>$e$</td>
<td>$r^2$</td>
</tr>
<tr>
<td>$v$</td>
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<td>$d_1$</td>
<td>$h$</td>
<td>$d_2$</td>
<td>$r$</td>
<td>$r^3$</td>
<td>$r^2$</td>
<td>$e$</td>
</tr>
</tbody>
</table>
It is natural to identify the rigid motions of the square with elements of $S_4$ in the following way.

<table>
<thead>
<tr>
<th>Rigid Motion</th>
<th>Corresponding permutation form $S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>$r$</td>
<td>$(1, 2, 3, 4)$</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$(1, 3)(2, 4)$</td>
</tr>
<tr>
<td>$r^3$</td>
<td>$(1, 4, 3, 2)$</td>
</tr>
<tr>
<td>$d_1$</td>
<td>$(2, 4)$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$(1, 3)$</td>
</tr>
<tr>
<td>$h$</td>
<td>$(1, 4)(2, 3)$</td>
</tr>
<tr>
<td>$v$</td>
<td>$(1, 2)(3, 4)$</td>
</tr>
</tbody>
</table>
Example

The 24 permutations on \( \{1, 2, 3, 4\} \) are

\[
S_4 = \left\{ e, (1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (1,4,3,2), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,2,4), (1,3,2), (1,3,4), (1,4,2), (1,4,3), (2,3,4), (2,4,3), (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \right\}
\]
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Feel free to write out the Cayley table.
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(1,4)(2,3), (1,2,3), (1,2,4), (1,3,2), (1,3,4), \\
(1,4,2), (1,4,3), (2,3,4), (2,4,3), (1,2), (1,3), \\
(1,4), (2,3), (2,4), (3,4) \} \]

Feel free to write out the Cayley table.
Compare this to the rigid motions of a square....
In our previous examples we saw:

1. two groups which although not equal seem “the same as groups”, and
2. a group which naturally “sits inside another group”.

In order to make these notions precise, we would like to consider maps which preserve group structure.
Note
In our previous examples we saw:

1. two groups which although not equal seem “the same as groups”, and
2. a group which naturally “sits inside another group”.

In order to make these notions precise, we would like to consider maps which preserve group structure.

Definition
Suppose that \((G, *)\) and \((H, \circ)\) are groups. A homomorphism from \(G\) to \(H\) is a map \(\phi : G \to H\) satisfying

\[
\phi(x * y) = \phi(x) \circ \phi(y),
\]

for all \(x, y \in G\).
Example

Let $G = \{\pm 1, \pm i\}$ and define $\phi : G \to \mathbb{Z}_4$ as follows

\[
\begin{align*}
\phi(1) &= 0 \\
\phi(i) &= 1 \\
\phi(-1) &= 2 \\
\phi(-i) &= 3
\end{align*}
\]

Check that $\phi$ is a homomorphism.
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Check that $\phi$ is a homomorphism.

Note that $\phi$ is also bijective.
Definition

Suppose that \((G, \ast)\) and \((H, \circ)\) are groups and that \(\phi : G \rightarrow H\) is a homomorphism.

- If \((G, \ast) = (H, \circ)\) then \(\phi\) is called an **endomorphism**.
- If \(\phi\) is surjective then it is called an **epimorphism**.
- If \(\phi\) is injective then it is called a **monomorphism**.
- If \(\phi\) is bijective then it is called a **isomorphism**.
- If \(\phi\) is bijective and \((G, \ast) = (H, \circ)\) then it is called an **automorphism**.
We define $\phi : D_4 \rightarrow S_4$ by

\[
\begin{align*}
\phi(e) &= e \\
\phi(r) &= (1, 2, 3, 4) \\
\phi(r^2) &= (1, 3)(2, 4) \\
\phi(r^3) &= (1, 4, 3, 2) \\
\phi(d_1) &= (2, 4) \\
\phi(d_2) &= (1, 3) \\
\phi(h) &= (1, 4)(2, 3) \\
\phi(v) &= (1, 2)(3, 4)
\end{align*}
\]

Here $\phi$ is a group monomorphism.
**Definition**

If $G$ and $H$ are groups and if there exists an isomorphism $\phi : G \rightarrow H$, then we say that $G$ and $H$ are isomorphic and write $G \cong H$. 

**Fact**

If $G$ is a set of groups then $\sim$ is an equivalence relation on $G$. 

**Note**

We have seen that $G = \{ \pm 1, \pm i \}$ is isomorphic to $\mathbb{Z}_4$. 

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MTHSC 412 Section 3.5 - 3.6 – Homomorphisms and Isomorphisms
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We have seen that $G = \{\pm 1, \pm i\}$ is isomorphic to $\mathbb{Z}_4$. 
Theorem

Suppose that $G$ and $H$ are groups and that $\phi : G \rightarrow H$ is a homomorphism. Then

1. $\phi(e_G) = e_H$
2. For all $g \in G$, $\phi(g^{-1}) = [\phi(g)]^{-1}$
**Example**

Define $\phi : \mathbb{Z} \to \mathbb{Z}_n$ by $\phi(x) = [x]$. 

Show that $\phi$ is a epimorphism.

**Definition**

If there exists an epimorphism $\phi : G \to H$, then $H$ is called a homomorphic image of $G$.

**Example**

$\mathbb{Z}_n$ is a homomorphic image of $\mathbb{Z}$. 

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MTHSC 412 Section 3.5 - 3.6 – Homomorphisms and Isomorphisms
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Let $\phi : G \to H$ be a homomorphism. The *kernel* of $\phi$ is defined by

$$\ker(\phi) = \{ g \in G \mid \phi(g) = e_H \}.$$
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**Example**

Again consider $\phi : \mathbb{Z} \to \mathbb{Z}_n$ defined by $\phi(x) = [x]$. 

**Fact**

Suppose that $\phi : G \to H$ be a homomorphism. Then $\ker(\phi) \leq G$. 
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Show that $\ker(\phi) = \{ nk \mid k \in \mathbb{Z} \}$. 

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**Fact**

*Suppose that $\phi : G \rightarrow H$ be a homomorphism. Then $\ker(\phi) \leq G.$*
Consider $\phi : \mathbb{Z} \rightarrow (\mathbb{R} - \{0\})$ defined by

$$\phi(x) = \begin{cases} 
1 & \text{if } x \text{ is even,} \\
-1 & \text{if } x \text{ is odd.}
\end{cases}$$

Show that $\phi$ is a homomorphism and compute its kernel.