MTHSC 412 Section 3.5 - 3.6 – Homomorphisms and Isomorphisms

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i	i	$i^2 = -1$	$i^3 = -i$	1
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$i^3 = -i$	$i^3 = -i$	1	i	$i^2 = -1$

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Do you notice similarities? Are these the same in some sense?

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- These functions form a group under composition of functions.
- The Cayley table for this group which is denoted D_4 is

$\circ$	e	r	r^2	r^3	d_1	d_2	h	v
e	e	r	r^2	r^3	d_1	d_2	h	v
r	r	r^2	r^3	e	v	h	d_1	d_2
r^2	r^2	r^3	e	r	d_2	d_1	v	h
r^3	r^3	e	r	r^2	h	v	d_2	d_1
d_1	d_1	h	d_2	v	e	r^2	r	r^3
d_2	d_2	v	d_1	h	r^2	e	r^3	r
h	h	d_2	v	d_1	r^3	r	e	r^2
v	v	d_1	h	d_2	r	r^3	r^2	e

Note

It is natural to identify the rigid motions of the square with elements of S_4 in the following way.

Rigid Motion	Corresponding permutation form S_4
e	e
r	$(1,2,3,4)$
r^2	$(1,3)(2,4)$
r^3	$(1,4,3,2)$
d_1	$(2,4)$
d_2	$(1,3)$
h	$(1,4)(2,3)$
v	$(1,2)(3,4)$

Example

The 24 permutations on $\{1,2,3,4\}$ are

$$S_4 = \left\{\begin{array}{l} \text{e, (1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2),} \\ \text{(1,4,2,3), (1,4,3,2), (1,2)(3,4), (1,3)(2,4),} \\ \text{(1,4)(2,3), (1,2,3), (1,2,4), (1,3,2), (1,3,4),} \\ \text{(1,4,2), (1,4,3), (2,3,4), (2,4,3) (1,2), (1,3),} \\ \text{(1,4), (2,3), (2,4), (3,4)} \end{array}\right\}$$

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Feel free to write out the Cayley table.

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Feel free to write out the Cayley table.
Compare this to the rigid motions of a square....

Note

In our previous examples we saw:

1. two groups which although not equal seem "the same as groups", and
2. a group which naturally "sits inside another group".

In order to make these notions precise, we would like to consider maps which preserve group structure.

Note

In our previous examples we saw:

1. two groups which although not equal seem "the same as groups", and
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In order to make these notions precise, we would like to consider maps which preserve group structure.

Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups. A *homomorphism* from G to H is a map $\phi : G \to H$ satisfying

$$\phi(x * y) = \phi(x) \circ \phi(y),$$

for all $x, y \in G$.

Example

Let $G = \{\pm 1, \pm i\}$ and define $\phi : G \to \mathbb{Z}_4$ as follows

$$\begin{aligned} \phi(1) &= 0 \\ \phi(i) &= 1 \\ \phi(-1) &= 2 \\ \phi(-i) &= 3 \end{aligned}$$

Check that ϕ is a homomorphism.

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Check that ϕ is a homomorphism.
Note that ϕ is also bijective.

Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups and that $\phi : G \to H$ is a homomorphism.

- If $(G, *) = (H, \circ)$ then ϕ is called an *endomorphism*.
- If ϕ is sujective then it is called an *epimorphism*.
- If ϕ is injective then it is called a *momomorphism*.
- If ϕ is bijective then it is called a *isomorphism*.
- If ϕ is bijective and $(G, *) = (H, \circ)$ then it is called an *automorphism*

Example

We define $\phi : D_4 \to S_4$ by

$$\begin{aligned}
\phi(e) &= e \\
\phi(r) &= (1,2,3,4) \\
\phi(r^2) &= (1,3)(2,4) \\
\phi(r^3) &= (1,4,3,2) \\
\phi(d_1) &= (2,4) \\
\phi(d_2) &= (1,3) \\
\phi(h) &= (1,4)(2,3) \\
\phi(v) &= (1,2)(3,4)
\end{aligned}$$

Here ϕ is a group monomorphism.

Definition

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Fact

If $\mathcal{G}$ is a set of groups then $\cong$ is an equivalence relation on $\mathcal{G}$.

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Note

We have seen that $G = \{\pm 1, \pm i\}$ is isomorphic to $\mathbb{Z}_4$.

Theorem

Suppose that G and H are groups and that $\phi : G \to H$ is a homomorphism. Then

1. $\phi(e_G) = e_H$
2. *Fro all* $g \in G$, $\phi(g^{-1}) = [\phi(g)]^{-1}$

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Example

$\mathbb{Z}_n$ is a homomorphic image of $\mathbb{Z}$.

Definition

Let $\phi : G \to H$ be a homomorphism. The *kernel* of ϕ is defined by

$$\ker(\phi) = \{g \in G \quad | \quad \phi(g) = e_H\}.$$

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Show that $\ker(\phi) = \{nk \quad | \quad k \in \mathbb{Z}\}$.

Fact

Suppose that $\phi : G \to H$ *be a homomorphism. Then* $\ker(\phi) \leq G$.

Example

Consider $\phi : \mathbb{Z} \to (\mathbb{R} - \{0\})$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

Show that ϕ is a homomorphism and compute its kernel.