MTHSC 412 Section 3.5 - 3.6 – Homomorphisms and Isomorphisms

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Example

Consider the Cayley tables for $G = \{\pm 1, \pm i\} =< i >$ and $\mathbb{Z}_4$.

$\cdot$	1	i	$i^2 = -1$	$i^3 = -i$
1	1	i	$i^2 = -1$	$i^3 = -i$
i	i	$i^2 = -1$	$i^3 = -i$	1
$i^2 = -1$	$i^2 = -1$	$i^3 = -i$	1	i
$i^3 = -i$	$i^3 = -i$	1	i	$i^2 = -1$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Do you notice similarities? Are these the same in some sense?

- There are 4 reflections (2 through the diagonals and 2 through lines bisecting opposite sides). We will denote these by d_1, d_2, v and h.
- There are 4 rotations through $0, \pi/2, \pi$ and $3\pi/2$ radians. Take r to be the rotation through $\pi/4$ then the rotations are e, r, r^2 and r^3.
- These functions form a group under composition of functions.
- The Cayley table for this group which is denoted D_4 is

$\circ$	e	r	r^2	r^3	d_1	d_2	h	v
e	e	r	r^2	r^3	d_1	d_2	h	v
r	r	r^2	r^3	e	v	h	d_1	d_2
r^2	r^2	r^3	e	r	d_2	d_1	v	h
r^3	r^3	e	r	r^2	h	v	d_2	d_1
d_1	d_1	h	d_2	v	e	r^2	r	r^3
d_2	d_2	v	d_1	h	r^2	e	r^3	r
h	h	d_2	v	d_1	r^3	r	e	r^2
v	v	d_1	h	d_2	r	r^3	r^2	e

Note

It is natural to identify the rigid motions of the square with elements of S_4 in the following way.

Rigid Motion	Corresponding permutation form S_4
e	e
r	$(1, 2, 3, 4)$
r^2	$(1, 3)(2, 4)$
r^3	$(1, 4, 3, 2)$
d_1	$(2, 4)$
d_2	$(1, 3)$
h	$(1, 4)(2, 3)$
v	$(1, 2)(3, 4)$

Example

The 24 permutations on $\{1, 2, 3, 4\}$ are

$$S_4 = \left\{ \begin{array}{l} e,\ (1,2,3,4),\ (1,2,4,3),\ (1,3,2,4),\ (1,3,4,2), \\ (1,4,2,3),\ (1,4,3,2),\ (1,2)(3,4),\ (1,3)(2,4), \\ (1,4)(2,3),\ (1,2,3),\ (1,2,4),\ (1,3,2),\ (1,3,4), \\ (1,4,2),\ (1,4,3),\ (2,3,4),\ (2,4,3)\ (1,2),\ (1,3), \\ (1,4),\ (2,3),\ (2,4),\ (3,4) \end{array} \right\}$$

Feel free to write out the Cayley table.
Compare this to the rigid motions of a square....

Note

In our previous examples we saw:

1. two groups which although not equal seem "the same as groups", and
2. a group which naturally "sits inside another group".

In order to make these notions precise, we would like to consider maps which preserve group structure.

Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups. A *homomorphism* from G to H is a map $\phi : G \to H$ satisfying

$$\phi(x * y) = \phi(x) \circ \phi(y),$$

for all $x, y \in G$.

Example

Let $G = \{\pm 1, \pm i\}$ and define $\phi : G \to \mathbb{Z}_4$ as follows

$$\begin{aligned} \phi(1) &= 0 \\ \phi(i) &= 1 \\ \phi(-1) &= 2 \\ \phi(-i) &= 3 \end{aligned}$$

Check that ϕ is a homomorphism.
Note that ϕ is also bijective.

Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups and that $\phi : G \rightarrow H$ is a homomorphism.

- If $(G, *) = (H, \circ)$ then ϕ is called an *endomorphism*.
- If ϕ is sujective then it is called an *epimorphism*.
- If ϕ is injective then it is called a *momomorphism*.
- If ϕ is bijective then it is called a *isomorphism*.
- If ϕ is bijective and $(G, *) = (H, \circ)$ then it is called an *automorphism*

Example

We define $\phi : D_4 \longrightarrow S_4$ by

$$
\begin{aligned}
\phi(e) &= e \\
\phi(r) &= (1,2,3,4) \\
\phi(r^2) &= (1,3)(2,4) \\
\phi(r^3) &= (1,4,3,2) \\
\phi(d_1) &= (2,4) \\
\phi(d_2) &= (1,3) \\
\phi(h) &= (1,4)(2,3) \\
\phi(v) &= (1,2)(3,4)
\end{aligned}
$$

Here ϕ is a group monomorphism.

Definition

If G and H are groups and if there exists an isomorphism $\phi : G \to H$, then we say that G and H are isomorphic and write $G \cong H$.

Fact

If $\mathcal{G}$ is a set of groups then $\cong$ is an equivalence relation on $\mathcal{G}$.

Note

We have seen that $G = \{\pm 1, \pm i\}$ is isomorphic to $\mathbb{Z}_4$.

Theorem

Suppose that G and H are groups and that $\phi : G \to H$ is a homomorphism. Then

1. $\phi(e_G) = e_H$
2. *Fro all* $g \in G$, $\phi(g^{-1}) = [\phi(g)]^{-1}$

Example

Define $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ by $\phi(x) = [x]$.
Show that ϕ is a epimorphism.

Definition

If there exists an epimorphism $\phi : G \rightarrow H$ then H is called a *homomorphic image* of G.

Example

$\mathbb{Z}_n$ is a homomorphic image of $\mathbb{Z}$.

Definition

Let $\phi : G \to H$ be a homomorphism. The *kernel* of ϕ is defined by

$$\ker(\phi) = \{g \in G \quad | \quad \phi(g) = e_H\}.$$

Example

Again consider $\phi : \mathbb{Z} \to \mathbb{Z}_n$ defined by $\phi(x) = [x]$.
Show that $\ker(\phi) = \{nk \quad | \quad k \in \mathbb{Z}\}$.

Fact

Suppose that $\phi : G \to H$ *be a homomorphism. Then* $\ker(\phi) \leq G$.

Example

Consider $\phi : \mathbb{Z} \to (\mathbb{R} - \{0\})$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

Show that ϕ is a homomorphism and compute its kernel.