MTHSC 412 Section 4.4 – Cosets of a subgroup

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Definition

Suppose that $(G, *)$ is a group and $A, B \subseteq G$. Then we define $A * B$ (or simply AB) by

$$AB = \{x \in G \quad | \quad x = ab \text{ for some } a \in A \text{ and } b \in B\}$$

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Example

Consider $G = S_4$, $A = \{e, (1,2,3), (1,3,2)\}$ and $B = \{(1,2), (2,3,4)\}$ Then,

$$AB = \{(1,2), (1,3), (2,3), (2,3,4), (1,2)(3,4), (1,3,4)\},$$

and

$$BA = \{(1,2), (2,3), (1,3), (2,3,4), (1,3)(2,4), (1,4,2)\}.$$

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Suppose that G is a group and $g \in G$ and $A \subseteq G$. Then we denote by gA and Ag the products $\{g\}A$ and $A\{g\}$.

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Theorem (Properties of products of subsets)

1. $A(BC) = (AB)C$.
2. $B = C \Rightarrow AB = AC$ *and* $BA = CA$
3. *In general* AB *and* BA *may be different.*
4. $AB = AC \not\Rightarrow B = C$
5. $gA = gB \Rightarrow A = B$.

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$(1,2,3)H = H$.
In fact, these are the only left cosets of H.

Lemma

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Thus $a = bkh^{-1} \in bH$ and $b = ahk^{-1} \in aH$.

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Thus $aH \subseteq bH$.
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Thus we have shown that $aH \cap bH \neq \emptyset \Rightarrow aH = bH$ and thus the distinct left cosets are pairwise disjoint. □

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Suppose that $G = S_3$ and $H = \{e, (1,2,3), (1,3,2)\}$.
We saw earlier that there are 2 distinct cosets of H in G.
So, $[G : H] = 2$.

Theorem (Lagrange's Theorem)

If $H \leq G$ and if G is finite, then

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Suppose that G is a group and that $|G| = p$ is prime. Then G is cyclic.