MTHSC 412 Section 4.4 – Cosets of a subgroup

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Definition

Suppose that $(G, *)$ is a group and $A, B \subseteq G$. Then we define $A * B$ (or simply AB) by

$$AB = \{x \in G \quad | \quad x = ab \text{ for some } a \in A \text{ and } b \in B\}$$

Example

Consider $G = S_4$, $A = \{e, (1,2,3), (1,3,2)\}$ and $B = \{(1,2), (2,3,4)\}$ Then,

$$AB = \{(1,2), (1,3), (2,3), (2,3,4), (1,2)(3,4), (1,3,4)\},$$

and

$$BA = \{(1,2), (2,3), (1,3), (2,3,4), (1,3)(2,4), (1,4,2)\}.$$

Notation

Suppose that G is a group and $g \in G$ and $A \subseteq G$. Then we denote by gA and Ag the products $\{g\}A$ and $A\{g\}$.

Theorem (Properties of products of subsets)

1. $A(BC) = (AB)C$.
2. $B = C \Rightarrow AB = AC$ *and* $BA = CA$
3. *In general* AB *and* BA *may be different.*
4. $AB = AC \not\Rightarrow B = C$
5. $gA = gB \Rightarrow A = B$.

Definition

Suppose that G is a group and that $H \leq G$. For $a \in G$ the set aH is a *left coset of* H. Similarly, the set Ha is called a *right coset of* H.

Example

Consider $G = S_3$ and $H = \{e, (1,2,3), (1,3,2)\}$. Then
$(1,2)H = \{(1,2), (2,3), (1,3)\}$
$(1,2,3)H = H$.
In fact, these are the only left cosets of H.

Lemma

Suppose that $H \leq G$. The distinct left cosets of H form a partition of G.

Proof.

Since $e \in H$, $a \in aH$ for all $a \in G$.
So, the left cosets of H are nonempty and their union is G.
Now suppose that $aH \cap bH \neq \emptyset$.
Let $c \in aH \cap bH$.
Then there is $h, k \in H$ such that $ah = c = bk$
Thus $a = bkh^{-1} \in bH$ and $b = ahk^{-1} \in aH$.
If $ah' \in aH$ then $ah' = bkh^{-1}h' \in bH$.
Thus $aH \subseteq bH$.
Similarly, $bH \subseteq aH$. Thus $aH = bH$.
Thus we have shown that $aH \cap bH \neq \emptyset \Rightarrow aH = bH$ and thus the distinct left cosets are pairwise disjoint. $\square$

Definition

Suppose that $H \leq G$. The *index* of H in G is defined to be the number of distinct left cosets of H in G and is denoted by $[G : H]$.

Example

Suppose that $G = S_3$ and $H = \{e, (1,2,3), (1,3,2)\}$.
We saw earlier that there are 2 distinct cosets of H in G.
So, $[G : H] = 2$.

Theorem (Lagrange's Theorem)

If $H \leq G$ and if G is finite, then

$$|G| = [G : H]|H|.$$

Corollary

Suppose that G is a finite group and that $g \in G$. Then $o(g)|\#G$.

Corollary

Suppose that G is a group and that $|G| = p$ is prime. Then G is cyclic.