MTHSC 412 Section 4.6 – Quotient Groups

Kevin James

Theorem

Let $H \trianglelefteq G$. Then the cosets of H form a group with the operation being the multiplication of sets. This group is denoted G/H.

Example

1. Note that $n\mathbb{Z} \trianglelefteq \mathbb{Z}$. What is $\mathbb{Z}/n\mathbb{Z}$?
2. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternian group. Note that $< i >= \{\pm 1, \pm i\} \trianglelefteq G$. What is $G/ < i >$?

Note

If $H \leq G$ and $[G : H] = 2$ then $H \trianglelefteq G$.

Theorem

Let $H \trianglelefteq G$. The mapping $\phi : G \to G/H$ given by $\phi(a) = aH$ is an epimorphism.

Theorem

For any homomorphism $\phi : G \to H$, $\ker(\phi) \trianglelefteq G$.

Theorem (Fundamental Theorem of Homomorphisms)

Suppose G and H are groups with H a homomorphic image of G (that is there is an epimorphism $\phi : G \to H$). Then

$$H \cong G/\ker(\phi).$$

Example

Let $G = S_3$ and let $H = U_3 = \{\pm 1\}$.
Define $\phi : G \to H$ by
$\phi(e) = \phi((1,2,3)) = \phi((1,3,2)) = 1$ and
$\phi((1,2)) = \phi((1,3)) = \phi((2,3)) = -1$.
Show that ϕ is an epimorphism.

Example (Solution)

First we make a few observations. Let $\sigma = (1, 2, 3)$ and $\tau = (1, 2)$. Then,

1. $S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$.
2. Thus any element of S_3 can be written as $\sigma^j\tau^k$ with $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$.
3. $\tau\sigma^i = \sigma^{-i}\tau$.
4. Note that $\phi(\sigma^j) = 1$ and $\phi(\sigma^j\tau) = -1$.

Solution continued

Now suppose that $x, y \in S_3$. Then,
$x = \sigma^j \tau^k$ with $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$, and
$y = \sigma^m \tau^n$ with $m \in \{0, 1, 2\}$ and $n \in \{0, 1\}$.
Case 1: If $k = 0$, then we have
$\phi(xy) = \phi(\sigma^j \sigma^m \tau^n) = \phi(\sigma^{j+m} \tau^n) = (-1)^n$
$\phi(x)\phi(y) = \phi(\sigma^j)\phi(\sigma^m \tau^n) = 1 \cdot (-1)^n$.
Case 2: If $k = 1$, then we have
$\phi(xy) = \phi(\sigma^j \tau \sigma^m \tau^n) = \phi(\sigma^{j-m} \tau^{n+1}) = (-1)^{n+1}$
$\phi(x)\phi(y) = \phi(\sigma^j \tau)\phi(\sigma^m \tau^n) = -1 \cdot (-1)^n = (-1)^{n+1}$.
Thus in either case $\phi(xy) = \phi(x)\phi(y)$ and ϕ is a homomorphism.
From our definition of ϕ, it is clear that ϕ is onto. So, ϕ is an epimorphism.

Example (continued)

In our previous example, we have $\ker(\phi) =< \sigma >$.
So, $G/\ker(\phi) \cong U_3$.
Letting $K =< \sigma >$, we have that $G/K = \{K, \tau K\}$.
Using ϕ we can define an isomorphism $\theta : G/K \to U_3$ by
$\theta(K) = \theta(eK) = \phi(e) = 1$ and
$\theta(\tau K) = \phi(\tau) = -1$.