MTHSC 412 Section 5.2 – Integral Domains and Fields

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Definition

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1. D is commutative.
2. D has a unity $1 \neq 0$.
3. D has no zero divisors.

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1. $\mathbb{Z}$ is an integral domain.
2. $\mathbb{Z}_{10}$ is not an integral domain, since it has zero divisors.

Theorem

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Theorem (Cancellation law for multiplication)

Suppose that D is an integral domain and that $a, b, c \in D$. Then $ab = ac \Rightarrow b = c$.

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Definition (Equivalent definition of a field)

A set F together with 2 binary operations $+$ and $\cdot$ is a field if the following conditions hold.

1. $(F, +)$ is an abelian group with identity denoted by 0.
2. $([F - \{0\}], \cdot)$ is an abelian group with identity denoted by 1.
3. $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Theorem

Every field is an integral domain.

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Every finite integral domain is a field.

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Example

Note that $M_n(\mathbb{Z})$, $M_n(\mathbb{Q})$, $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are not integral domains since

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$