# MTHSC 412 Section 5.2 – Integral Domains and Fields

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### Definition

A ring $D$ is an *integral domain* if the following conditions hold.

1. $D$ is commutative.
2. $D$ has a unity $1 \neq 0$.
3. $D$ has no zero divisors.

### Example

1. $\mathbb{Z}$ is an integral domain.
2. $\mathbb{Z}_{10}$ is not an integral domain, since it has zero divisors.

**Theorem**

*For $n > 1$, $\mathbb{Z}_n$ is an integral domain if and only if $n$ is prime.*

**Theorem (Cancellation law for multiplication)**

*Suppose that $D$ is an integral domain and that $a, b, c \in D$. Then $ab = ac \Rightarrow b = c$.*

**Definition**

A ring $F$ is a *field* if the following conditions hold.

1. $F$ is a commutative ring.
2. $F$ has a unity $1 \neq 0$
3. Every nonzero element of $F$ has a multiplicative inverse.

**Definition (Equivalent definition of a field)**

A set $F$ together with 2 binary operations $+$ and $\cdot$ is a field if the following conditions hold.

1. $(F, +)$ is an abelian group with identity denoted by 0.
2. $([F - \{0\}], \cdot)$ is an abelian group with identity denoted by 1.
3. $x(y + z) = xy + xz$ for all $x, y, z \in F$.

**Theorem**

*Every field is an integral domain.*

### Theorem

*Every finite integral domain is a field.*

### Theorem

*$\mathbb{Z}_n$ is a field if and only if $n$ is prime.*

### Example

Note that $M_n(\mathbb{Z})$, $M_n(\mathbb{Q})$, $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are not integral domains since

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$