MTHSC 412 Section 6.1 – Ideals and Quotient Rings

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Definition

A subring I of a ring R is an ideal in R if for all $x \in I$ and $r \in R$, $xr, rx \in I$. If I is an ideal of R, we will write $I \trianglelefteq R$.

Note

If R is any ring, then $\{0\}$ and R are ideals of R. These are referred to as *trivial ideals*.

Example

Note that for $m \geq 1$, $m\mathbb{Z} \trianglelefteq \mathbb{Z}$.

Theorem

Suppose that $I \subseteq R$ where R is a ring. Then $I \trianglelefteq R$ if the following are true.

1. $I \neq \emptyset$.
2. $x, y \in I \Rightarrow (x + y) \in I$.
3. $x \in I \Rightarrow -x \in I$.
4. $x \in I; r \in R \Rightarrow xr, rx \in I$.

Example

Note that $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ is a ring.

Let $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b, \in \mathbb{Z} \right\}$.

Show that $I \trianglelefteq S$.

Definition

Suppose that R is a commutative ring with 1 and that $a \in R$. The *principal ideal generated by a* is defined by

$$< a >= \{ar \mid r \in R\}$$

Theorem

Suppose that R is a commutative ring with 1 and $a \in R$. Then $< a > \trianglelefteq R$.

Theorem ($\mathbb{Z}$ is a Principal Ideal Domain)

In the ring $\mathbb{Z}$, every ideal is principal.

Definition

Suppose that R is a ring and $I \trianglelefteq R$. Then we consider the cosets

$$r + I = \{r + k \mid k \in I\}$$

and define addition and multiplication of these cosets as follows.

$$(a + I) + (b + I) = (a + b) + I; \quad (a + I)(b + I) = ab + I.$$

Theorem

Suppose that R is a ring and $I \trianglelefteq R$. Then $R/I = \{r + I \mid r \in R\}$ is a ring with addition and multiplication defined as above. R/I is called the quotient ring *of R by I.*

Example

Consider $\mathbb{Z}/4\mathbb{Z}$.