MTHSC 412 Section 6.2 – Ring Homomorphisms

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Definition

If R and S are rings a map $\phi : R \to S$ is a *ring homomorphism* if for all $m, n \in R$,

1. $\phi(m + n) = \phi(m) + \phi(n)$ and,
2. $\phi(mn) = \phi(m)\phi(n)$.

We also define ring monomorphisms, epimorphisms, isomorphisms, endomorphisms and automorphisms as with groups.

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Example

Define $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ by $\phi(x) = 2x$. Check that ϕ is a ring endomorphism. Is it injective, surjective?

Theorem

Suppose that $\phi : R \rightarrow S$ is a ring homomorphism. Then,

1. $\phi(0) = 0$.
2. $\phi(-r) = -\phi(r)$.

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Theorem

Suppose that $\phi : R \rightarrow S$ is a ring homomorphism.

1. *If T is a subring of R then $\phi(T)$ is a subring of S.*
2. *If V is a subring of S then $\phi^{-1}(V)$ is a subring of R.*

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Suppose that $\phi : R \to S$ is a ring homomorphism. Then we define the *kernel* of ϕ as

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Theorem

Suppose that $\phi : R \to S$ is a ring homomorphism. Then $\ker(\phi) \trianglelefteq R$. and $\ker(\phi) = \{0\}$ if and only if ϕ is injective.

Theorem

Suppose that $I \trianglelefteq R$. *Define* $\theta : R \longrightarrow R/I$ *by* $\theta(r) = r + I$. *Then* θ *is an epimorphism and* $\ker(\theta) = I$.

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Theorem

Suppose that $\phi : R \longrightarrow S$ *is an epimorphism. Then* $S \cong R/\ker(\phi)$.