MTHSC 412 Section 6.2 – Ring Homomorphisms

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Definition

If R and S are rings a map $\phi : R \to S$ is a *ring homomorphism* if for all $m, n \in R$,

1. $\phi(m+n) = \phi(m) + \phi(n)$ and,
2. $\phi(mn) = \phi(m)\phi(n)$.

We also define ring monomorphisms, epimorphisms, isomorphisms, endomorphisms and automorphisms as with groups.

Example

Define $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ by $\phi(x) = 2x$. Check that ϕ is a ring endomorphism. Is it injective, surjective?

Theorem

Suppose that $\phi : R \to S$ *is a ring homomorphism. Then,*

1. $\phi(0) = 0.$
2. $\phi(-r) = -\phi(r).$

Theorem

Suppose that $\phi : R \to S$ *is a ring homomorphism.*

1. *If* T *is a subring of* R *then* $\phi(T)$ *is a subring of* S*.*
2. *If* V *is a subring of* S *then* $\phi^{-1}(V)$ *is a subring of* R*.*

Definition

Suppose that $\phi : R \to S$ is a ring homomorphism. Then we define the *kernel* of ϕ as

$$\ker(\phi) = \{r \in R \mid \phi(r) = 0\}.$$

Theorem

Suppose that $\phi : R \to S$ is a ring homomorphism. Then $\ker(\phi) \trianglelefteq R$. and $\ker(\phi) = \{0\}$ if and only if ϕ is injective.

Theorem

Suppose that $I \trianglelefteq R$. *Define* $\theta : R \to R/I$ *by* $\theta(r) = r + I$. *Then* θ *is an epimorphism and* $\ker(\theta) = I$.

Theorem

Suppose that $\phi : R \to S$ *is an epimorphism. Then* $S \cong R/\ker(\phi)$.