MTHSC 412 Section 1.3 – Primes and Unique Factorization

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Definition

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Example

2, 3, 5, 7, 11, 13, 17, 19, 23 and 29 are primes.
4, 6, 8, 9, 10, 12 are not.

Fact

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- *If p and q are prime and $p|q$, then $p = \pm q$.*

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Theorem

Let $0, \pm 1 \neq p \in \mathbb{Z}$. Then p is a prime if and only if whenever $p|bc$, $p|b$ or $p|c$.

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Also, $d|b$.

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Also, $d|b$. So, $b = dk$ for some $k \in \mathbb{Z}$.

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Case 2: $d = 1$. In this case, we have $p|bc$ and $(p, b) = 1$.

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($\Leftarrow$): Exercise. $\square$

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By the previous theorem, we can conclude that $p | a_1$ or $p | (a_2 \cdot \cdots \cdot a_n)$.
In the latter case, we can again apply the theorem, to conclude that either $p | a_2$ or $p | (a_3 \cdot \cdots \cdot a_n)$.

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In the latter case, we can again apply the theorem, to conclude that either $p|a_2$ or $p|(a_3\cdot\dots\cdot a_n)$.
Thus after at most $n-1$ applications or our theorem, we can conclude that $p|a_i$ for some $1\leq i\leq n$. $\square$

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Note

We allow expressions involving only one prime. Thus any prime is easily expressible as a product of primes.

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We will show that $S = \emptyset$.

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Assume for the sake of contradiction that $S \neq \emptyset$.
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Since $m \in S$, m cannot be a prime.
Thus m has some divisor other than $\pm 1, \pm m$, say a.
We may assume that $a > 1$ and thus that $m = ab$ where $1 < a, b < m$ and $a, b \in \mathbb{Z}$.
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Thus, $m = ab = p_1 \cdot \cdots \cdot p_t \cdot q_1 \cdot \cdots \cdot q_r$ which is a product of primes.

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So, we conclude that $S = \emptyset$ and thus all integers greater than 1 can be expressed as a product of primes, and as mentioned above this implies that all integers other than 0 and ± 1 can be expressed as a product of primes. $\square$

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Every $0, \pm 1 \neq n \in \mathbb{Z}$ *can be written as a product of primes. This factorization is unique up to rearrangement and sign change.*

Corollary

Every integer $n > 1$ *can be expressed uniquely as* $n = p_1 \cdot \cdots \cdot p_t$ *where the* $p_1 \leq p_2 \leq \cdots \leq p_t$ *and where the* p_i*'s are positive primes.*

Theorem

Let $n > 1$. If n has no positive prime factor less than or equal to $\sqrt{n}$, then n is prime.