MTHSC 412 Section 2.1 –Congruence and Congruence Classes

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Definition

Let $a, b, n \in \mathbb{Z}$ with $n > 0$. We say that <u>a is congruent to b modulo n</u> and write $a \equiv b \pmod{n}$ when $n | (a - b)$.

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Example

1. $1 \equiv 5 \pmod{4}$.
2. $2 \equiv 17 \pmod{3}$.
3. $16 \equiv 4 \pmod{3}$.
4. $1 \not\equiv 5 \pmod{3}$.

Theorem

Let $0 < n \in \mathbb{Z}$*. Then* $\equiv \pmod{n}$ *is an equivalence relation on* $\mathbb{Z}$*. That is,* $\equiv \pmod{n}$ *satisfies the following three properties.*

Reflexivity $x \equiv x \pmod{n}$ *for all* $x \in \mathbb{Z}$*.*

Symmetry *If* $x \equiv y \pmod{n}$ *then* $y \equiv x \pmod{n}$*.*

Transitivity *If* $x \equiv y \pmod{n}$ *and* $y \equiv z \pmod{n}$ *then* $x \equiv z \pmod{n}$

Proof.

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Then $n|(x - y) \Rightarrow (x - y) = nk$ for some $k \in \mathbb{Z}$.

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Then $n|(x-y) \Rightarrow (x-y) = nk$ for some $k \in \mathbb{Z}$.
So, $(y-x) = n(-k)$. Thus, $n|(y-x)$ and $y \equiv x \pmod{n}$.

PROOF.

Let $n > 0$ be an integer.
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Transitive: Suppose that $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$
Then $(x-y) = nk$ and $(y-z) = nm$ for some $k, m \in \mathbb{Z}$.

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So, $(x-z) = (x-y) + (y-z) = n(k+m)$ and $n|(x-z)$.

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Transitive: Suppose that $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$
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So, $(x - z) = (x - y) + (y - z) = n(k + m)$ and $n|(x - z)$.
Thus $x \equiv z \pmod{n}$ and $\equiv \pmod{n}$ is transitive. □

THEOREM

If $a \equiv b \pmod{n}$ *and* $x \in \mathbb{Z}$ *then*

$$a + x \equiv b + x \pmod{n} \quad \text{and} \quad ax \equiv bx \pmod{n}.$$

Addition and Multiplication Properties

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If $a \equiv b \pmod{n}$ and $x \in \mathbb{Z}$ then

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Suppose that $a \equiv b \pmod{n}$. Then $(a - b) = nk$ for some $k \in \mathbb{Z}$.
Thus $(a + x) - (b + x) = a - b = nk$

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Suppose that $a \equiv b \pmod{n}$. Then $(a - b) = nk$ for some $k \in \mathbb{Z}$.
Thus $(a + x) - (b + x) = a - b = nk$ and
$ax - bx = x(a - b) = xnk$ and the result follows. □

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Suppose that $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$. *Then*

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Substitution

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Thus $ac \equiv bd \pmod{n}$. □

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Example

Suppose $n = 5$. Then [9] is an infinite set which contains -6,-1,4,9,14,19 and 24.

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We have now shown that $[a] \subseteq [c]$.

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Similarly, we can show that $[c] \subseteq [a]$ and so we can conclude that $[a] = [c]$.

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($\Leftarrow$): Suppose now that $[a] = [c]$.
By reflexivity, $a \in [a] = [c]$.

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Similarly, we can show that $[c] \subseteq [a]$ and so we can conclude that $[a] = [c]$.
($\Leftarrow$): Suppose now that $[a] = [c]$.
By reflexivity, $a \in [a] = [c]$.
Thus, $a \equiv c \pmod{n}$. □

Corollary

Suppose that $0 < n \in \mathbb{Z}$ *and* $a, c \in \mathbb{Z}$. *Either* $[a] \cap [c] = \emptyset$ *or* $[a] = [c]$.

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Let $x \in [a] \cap [c]$.
So, we have $x \equiv a \pmod{n}$ and $x \equiv c \pmod{n}$.
By symmetry, we have $a \equiv x \pmod{n}$ and $x \equiv c \pmod{n}$.

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Suppose that $0 < n \in \mathbb{Z}$ *and* $a, c \in \mathbb{Z}$. *Either* $[a] \cap [c] = \emptyset$ *or* $[a] = [c]$.

Proof.

Suppose that $0 < n \in \mathbb{Z}$ and $a, c \in \mathbb{Z}$ and that $[a] \cap [c] \neq \emptyset$.
Let $x \in [a] \cap [c]$.
So, we have $x \equiv a \pmod{n}$ and $x \equiv c \pmod{n}$.
By symmetry, we have $a \equiv x \pmod{n}$ and $x \equiv c \pmod{n}$.
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By transitivity we have $a \equiv c \pmod{n}$.
By our previous theorem, we now have that $[a] = [c]$. □

Corollary

Let $1 < n \in \mathbb{Z}$.

1. *If* $a \in \mathbb{Z}$ *and* $a = nq + r$ *(e.g.* r *could be the remainder produced when* a *is divided by* n*.), then* $[a] = [r]$.
2. *There are exactly* n *distinct congruence classes modulo* n*, namely* $[0], [1], \ldots, [n-1]$.

Proof.

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Then we can write $a = nq + r$ with $0 \leq r \leq (n - 1)$.

PROOF.

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So, it follows from part (1) that $[a] = [r]$.

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Then $-n < (i-j) < n$ and $i \equiv j \pmod{n} \Rightarrow$

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Suppose that $0 \leq i, j \leq (n-1)$ and $[i] = [j]$.
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Thus, $i - j = 0 \Rightarrow i = j$.
Therefore $[0], [1], \ldots, [n-1]$ are distinct. $\square$

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Note

From above we see that

$$\begin{aligned} \mathbb{Z}_n &= \{[a] \mid a \in \mathbb{Z}\} \\ &= \{[0], [1], [2], \ldots, [n-1]\}. \end{aligned}$$