MTHSC 412 Section 2.2 – Modular Arithmetic

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Definition

We define addition and multiplication on $\mathbb{Z}_n$ as follows.

$$[a] + [b] = [a + b] \qquad \text{and} \qquad [a][b] = [ab].$$

Note

We must check that this addition is well-defined since any congruence class can be represented as $[a]$ in many ways. For example, if $n = 5$ then $[1] = [6] = [11]$. In fact, for any $b \in [a]$, $[a] = [b]$.

Theorem

If $[a] = [b]$ *and* $[c] = [d]$, *then*

$$[a + c] = [b + d] \qquad \textit{and} \qquad [ac] = [bd].$$

Proof.

Suppose that $[a] = [b]$ and $[c] = [d]$.
Then $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
Thus from theorems of the previous section, $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$.
Thus, $[a + c] = [b + d]$ and $[ac] = [bd]$. □

Example

The addition table for $\mathbb{Z}_6$ is as follows.

$+$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[1]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$
$[2]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$
$[3]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$
$[4]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$
$[5]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$

Example

The multiplication table for $\mathbb{Z}_6$ is as follows.

$\times$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

Theorem

Suppose that $[a], [b], [c] \in \mathbb{Z}_n$.

1. *Closure of addition:* $[a] + [b] \in \mathbb{Z}_n$.
2. *Associativity of addition:* $([a] + [b]) + [c] = [a] + ([b] + [c])$.
3. *Commutativity of addition:* $[a] + [b] = [b] + [a]$.
4. *Additive Identity:* $[a] + [0] = [0] + [a] = [a]$.
5. *Additive Inverses:* $[a] + [-a] = [-a] + [a] = [0]$.
6. *Closure of Multiplication:* $[a][b] \in \mathbb{Z}_n$.
7. *Associativity of Multiplication:* $([a][b])[c] = [a]([b][c])$.
8. *Commutativity of Multiplication:* $[a][b] = [b][a]$.
9. *Distributive Laws:* $[a]([b] + [c]) = [a][b] + [a][c]$ *and* $([a] + [b])[c] = [a][c] + [b][c]$.
10. *Multiplicative Identity:* $[a][1] = [1][a] = [a]$.

Note

1. We will use positive integral exponents to denote repeated multiplication as usual.
2. $[a]^0 = [1]$.
3. Given $[a] \in \mathbb{Z}_n$, if there is $[b] \in \mathbb{Z}_n$ such that $[a][b] = [1]$, then we say that $[b]$ is the multiplicative inverse of $[a]$ and write $[b] = [a]^{-1}$.
4. For $n > 0$, $[a]^{-n} = ([a]^{-1})^n$.