MTHSC 412 Section 3.1 – Definition and Examples of Rings

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Definition

A ring R is a nonempty set R together with two binary operations (usually written as addition and multiplication) that satisfy the following axioms. Suppose that $a, b, c \in R$.

1. $a + b \in R$. (R is closed under addition.)
2. $a + (b + c) = (a + b) + c$. (Associativity of addition)
3. $a + b = b + a$. (Commutativity of addition)
4. There is an element $0_R \in R$ such that $a + 0 = 0 + a = a$. (Additive Identity or Zero element).
5. For each $a \in R$, the equation $a + x = 0_R$ has a solution in R, usually denoted $-a$. (Additive Inverses)
6. $ab \in R$. (R is closed under multiplication)
7. $a(bc) = (ab)c$. (Associativity of multiplication)
8. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$. (Distributive laws)

Definition

A commutative ring R is a ring which also satisfies

9. $ab = ba$, for all $a, b \in R$. (Commutativity of multiplication)

Definition

A ring with identity is a ring R that contains an element 1_R satisfying the following.

10. $1_R a = a 1_R = a$, for all $a \in R$. (Multiplicative Identity)

Example

1. $\mathbb{Z}$ with the usual definition of addition and multiplication is a commutative ring with identity.
2. $\mathbb{Z}_n$ with addition and multiplication as defined in chapter 2 is a commutative ring with identity.
3. The set E of even integers is a commutative ring (without identity).
4. The set O of odd integers is not a ring.
5. The set $T = \{r, s, t, z\}$ is a ring under the addition and multiplication defined below.

$+$	z	r	s	t
z	z	r	s	t
r	r	z	t	s
s	s	t	z	r
t	t	s	r	z

and

$\cdot$	z	r	s	t
z	z	z	z	z
r	z	z	r	r
s	z	z	s	s
t	z	z	t	t

Example

6. The set $\mathbb{M}_2(\mathbb{R})$ of 2×2 matrices with real entries is a (noncommutative) ring with identity.
7. Similarly, the sets $\mathbb{M}_2(\mathbb{Z}), \mathbb{M}_2(\mathbb{Z}_n), \mathbb{M}_2(\mathbb{Q}), \mathbb{M}_2(\mathbb{C})$ are (noncommutative) rings with identity.
8. $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ is a ring under the operations $fg(x) = f(x)g(x)$ and $(f+g)(x) = f(x) + g(x)$.

Definition

An integral domain is a commutative ring R with identity $1_R \neq 0_R$ that satisfies the following.

1.1 Whenever $a, b \in R$ and $ab = 0$, either $a = 0$ or $b = 0$.

Example

1. $\mathbb{Z}$ is an integral domain.
2. If p is prime, then $\mathbb{Z}_p$ is an integral domain.
3. $\mathbb{Q}$ is an integral domain.
4. $\mathbb{Z}_6$ is **NOT** and integral domain.

Definition

A <u>field</u> is a commutative ring R with identity $1_R \neq 0_R$. that satisfies the following condition.

12. For each $0_R \neq a \in R$, the equation $ax = 1_R$ has a solution in R.

Example

1. $\mathbb{Q}$ is a field.
2. $\mathbb{R}$ is a field
3. $\mathbb{C}$ is a field.
4. If p is prime then $\mathbb{Z}_p$ is a field.

Theorem

Let R and S be rings. Define addition and multiplication on $R \times S$ by

$$(a, b) + (c, d) = (a + c, b + d), \qquad \text{and } (a, b)(c, d) = (ac, bd).$$

Then $R \times S$ is a ring. If both R and S are commutative then so is $R \times S$. If both R and S have an identity then so does $R \times S$, namely $1_{R\times S} = (1_R, 1_S)$.

Proof.

Exercise. $\square$

Definition

If R is a ring and $S \subset R$ is also a ring under the same operations as R, the we say that S is a subring of R.

Example

1. $\mathbb{Z}$ is a subring of $\mathbb{Q}$.
2. $\mathbb{Q}$ is a subring of $\mathbb{R}$. In fact, we can say that $\mathbb{Q}$ is a subfield of $\mathbb{R}$.

Theorem

Suppose that R is a ring and that $S \subseteq R$ satisfies

1. *S is closed under addition.*
2. *S is closed under multiplication.*
3. $0_R \in S$.
4. *If $a \in S$ then the solution to $a + x = 0_R$ is in S.*

Then S is a subring of R.

Proof.

$\square$

Example

$\{0, 3\}$ is a subring of $\mathbb{Z}_6$.