MTHSC 412 Section 4.1 – Polynomial Arithmetic and the Division Algorithm

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Exercise

Prove that $R[x]$ is a ring if R is.

Note

1. We can think of $R \subset R[x]$.
2. $R[x]$ is commutative if and only if R is.
3. If R has an identity 1_R then $R[x]$ also has an identity, namely $1_{R[x]} = 1_R \cdot x^0$ (which we usually write as 1).

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Theorem

If R is an integral domain and $0_R \neq f, g \in R[x]$, then

$$\deg(fg) = \deg(f) + \deg(g).$$

Proof.

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PROOF.

Suppose that R is an integral domain and $0_R \neq f, g \in R[x]$. Write $f = \sum f_n x^n$ and $g = \sum g_n x^n$ and let d_f and d_g denote the degrees of f and g.

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Thus $\deg(fg) = d_f + d_g$. □

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Proof.

Suppose that $f \neq 0$ and $g \neq 0$ are elements in $R[x]$. Then f and g have some nonzero coefficients. Following the argument of the last proof, we see that fg will also have a nonzero coefficient (namely the coefficient of $x^{d_f+d_g}$), and is thus nonzero. So, $R[x]$ is an integral domain. $\square$

Note

By a the first part of the proof of our theorem, we have seen that if R is any ring and if $f, g \in R[x]$ are nonzero, then $\deg(fg) \leq \deg(f) + \deg(g)$.

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Example

In $\mathbb{Z}_6[x]$ we have,

$$(3x + 1)(2x + 1) = 5x + 1.$$

Theorem

Let F be a field and $f, g \in F[x]$ with $g \neq 0$. Then there exist polynomials $q, r \in F[x]$ satisfying the following.

1. $f = gq + r$.
2. *Either* $r = 0$ *or* $\deg(r) < \deg(g)$.