MTHSC 412 Section 4.2 – Divisibility in $F[x]$

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Definition

Suppose that F is a field and that $f, g \in F[x]$ with $f \neq 0$. We say that f <u>divides</u> g or is a <u>factor</u> of g and write $f|g$ if $g = fh$ for some $h \in F[x]$.

Note

1. If $f|g$ then for any $0 \neq c \in F$, $(cf)|g$.
2. If $f|g$, then $\deg(f) \leq \deg(g)$.

Definition

We say that $f \in F[x]$ is monic if the leading coefficient of f is 1.

Definition

Let F be a field and suppose that $f, g \in F[x]$ are not both zero. Then the greatest common divisor of f and g is the monic polynomial d satisfying

1. $d|f$ and $d|g$.
2. If $c|f$ and $c|g$, then $\deg(c) \leq \deg(d)$.

Theorem

Suppose that F is a field and that $f, g \in F[x]$ are not both zero. Then, there is a unique greatest common divisor d of f and g. Furthermore, there exist polynomials $u, v \in F[x]$ such that

$$d = fu + gv.$$

Corollary

Suppose that F is a field and that $f, g \in F[x]$ are not both zero. A monic polynomial $t \in F[x]$ is the greatest common divisor of f and g if and only if it satisfies the following condiitons.

1. $d | f$ and $d | g$.
2. if $c | f$ and $c | g$ then $c | d$.

Definition

Suppose that F is a field and that $f, g \in F$ are not both zero. If the gcd of f and g is 1, then we say that f and g are relatively prime or coprime.

Theorem

Suppose that F is a field and that $f, g, h \in F[x]$. If $f|(gh)$ and f and g are coprime, then $f|h$.

Note

Since we have a division algorithm for $F[x]$, the Euclidean algorithm can be adapted to $F[x]$ and this gives an efficient method for computing the gcd and for writing the gcd as a combination of f and g.