MTHSC 412 Section 4.3 – Irreducibles and Unique Factorization in $F[x]$

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Theorem

Let R be an integral domain. Then $f \in R[x]$ is a unit in $R[x]$ if and only f is a constant polynomial that is a unit in R.

Corollary

Let F be a field. Then f is a unit in $F[x]$ if and only if f is a nonzero constant polynomial (-i.e. if and only if $\deg(f) = 0$).

Definition

An element r in a commutative ring R with identity is said to be an associate of an element $b \in R$ if there is a unit $u \in R$ such that $a = bu$.

Example

3 and -3 are associates in $\mathbb{Z}$.

Definition

1. Let F be a field and suppose that $f \in F[x]$. The polynomial f is said to be irreducible if its only divisors are its associates and the nonzero constant polynomials (units of $F[x]$).
2. A nonconstant polynomial that is not irreducible is said to be reducible.

Fact

Every polynomial of degree 1 in $F[x]$ is irreducible.

Theorem

Let F be a field. A polynomial $f \in F[x]$ is reducible if and only if it can be written as a product of two polynomials of lower degree.

Theorem

Suppose that F is a field and that $p \in F[x]$ with $\deg(p) > 0$. Then the following are equivalent.

1. *p is irreducible.*
2. *If $b, c \in F[x]$ and $p|(bc)$ then either $p|b$ or $p|c$.*
3. *If $r, s \in F[x]$ are such that $p = rs$ then r or s is a nonzero constant polynomial.*

Corollary

Suppose that F is a field and that $p \in F[x]$ is irreducible. If $p|(a_1 \cdot a_2 \cdot \cdots \cdot a_k)$ then $p|a_i$ for some $1 \leq i \leq k$.

Theorem

Let F be a field. Every nonconstant polynomial $f \in F[x]$ can be written as a product of irreducible polynomials of $F[x]$. This factorization is unique up to rearrangement replacement by associates.

Corollary

Suppose that F is a field. Every $f \in F[x]$ with $\deg(f) > 0$ can be uniquely written as $f = u \cdot p_1 \cdot p_2 \cdot \cdots \cdot p_k$, where u is a unit of $F[x]$ and the p_i's are monic irreducibles of $F[x]$.