MTHSC 412 Section 4.4 –Polynomial Functions, Roots and Reducibility

Kevin James

Note

Suppose that R is a commutative ring. To each polynomial $f = \sum f_n x^n \in R[x]$ we can associate a function $f : R \to R$ defined by the rule $f(r) = \sum f_n r^n$.

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Note

Let $f(x) = x^5$ and $g(x) = x$ be polynomials in $\mathbb{Z}_5[x]$. Note that although f and g are different elements in $\mathbb{Z}_5[x]$, they induce the same function from $\mathbb{Z}_5$ to $\mathbb{Z}_5$.

Definition

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Example

Let $R = \mathbb{Z}_5$ and let $f = x^2 + 1$. Then it is easy to check that 2 and 3 are roots of f.

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Theorem (Factor Theorem)

Suppose that F is a field and that $f \in F[x]$. Then $a \in F$ is a root of f if and only if $(x - a)|f$.

COROLLARY

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Corollary

Let F be an infinite field and $f, g \in F[x]$. Then f and g induce the same function from F to F if and only if $f = g$.