MTHSC 412 Section 4.4 –Polynomial Functions, Roots and Reducibility

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NOTE

Suppose that R is a commutative ring. To each polynomial $f = \sum f_n x^n \in R[x]$ we can associate a function $f : R \to R$ defined by the rule $f(r) = \sum f_n r^n$.

NOTE

Let $f(x) = x^5$ and $g(x) = x$ be polynomials in $\mathbb{Z}_5[x]$. Note that although f and g are different elements in $\mathbb{Z}_5[x]$, they induce the same function from $\mathbb{Z}_5$ to $\mathbb{Z}_5$.

DEFINITION

Let R be a commutative ring and $f \in R[x]$. An element $a \in R$ is said to be a <u>root</u> of f if $f(a) = 0_R$.

EXAMPLE

Let $R = \mathbb{Z}_5$ and let $f = x^2 + 1$. Then it is easy to check that 2 and 3 are roots of f.

Theorem (Remainder Theorem)

Suppose that F is a field and that $f \in F[x]$ and $a \in F$. Then the reminder when f is divided by $(x - a)$ is $f(a)$.

Theorem (Factor Theorem)

Suppose that F is a field and that $f \in F[x]$. Then $a \in F$ is a root of f if and only if $(x - a)|f$.

COROLLARY

Suppose that F is a field and that $f \in F[x]$ has degree n. Then f has at most n roots in F.

COROLLARY

Let F be a field and let $f \in F[x]$ with degree ≥ 2. If f is irreducible then f has no roots in F.

COROLLARY

*Suppose that F is a field, $f \in F[x]$ and that $\deg(f) = 2$ or 3.
Then f is irreducible if and only if f has no roots in F.*

EXAMPLE

$x^2 + x + 1$ is irreducible in $\mathbb{Z}_5$.

COROLLARY

*Let F be an infinite field and $f, g \in F[x]$. Then f and g induce the
same function from F to F if and only if $f = g$.*