MTHSC 412 Section 5.1 –Congruence in $F[x]$ and Congruence Classes

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Definition

Suppose that F is a field and that $f, g, p \in F[x]$ with $p \neq 0$. Then we say that f *is congruent to* g *modulo* p and write $f \equiv g \pmod{p}$ if $p|(f-g)$.

Example

In $\mathbb{Q}[x]$, $x^3 + 3x^2 + 4x + 1 \equiv x - 1 \pmod{x^2 + x + 1}$

Theorem

Let F be a field and $0 \neq p \in F[x]$. Then congruence modulo p is an equivalence relation on $F[x]$.

Theorem

Let F be a field and $0 \neq p \in F[x]$. Suppose that $f \equiv g \pmod{p}$ and $h \equiv k \pmod{p}$. Then

$$f + h \equiv g + k \pmod{p} \qquad \text{and} \qquad fh \equiv gk \pmod{p}.$$

Definition

Suppose that F is a field and that $f, p \in F[x]$ with $p \neq 0$. We define the congruence class of f as

$$[f] = \{g \in F[x] \mid f \equiv g \pmod{p}\}.$$

Theorem

$f \equiv g \pmod{p}$ if and only if $[f] = [g]$.

Corollary

Two congruence classes modulo p are either identical or disjoint.

Corollary

Suppose that F is a field and that $0 \neq p \in F[x]$. Let

$$S = \{f \in F[x] : \deg(f) < \deg(p)\} \cup \{0\}.$$

Then, $\cup_{f \in S}[f] = F[x]$ and if $f, g \in S$ then $[f] = [g]$ if and only if $f = g$.

Example

Consider congruence modulo $x^2 + 1$ in $\mathbb{R}[x]$. What are the congruence classes. Which one is congruent to x^2?

Notation

We denote the set of congruence classes of $F[x]$ modulo p by $F[x]/(p)$.