MTHSC 412 Section 6.2 – Quotient Rings and Homomorphisms

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Theorem

Let $I \trianglelefteq R$. If $a + I = b + I$ and $c + I = d + I$, then

$$(a + c) + I = (b + d) + I \qquad \text{and} \qquad ac + I = bd + I.$$

Definition

Suppose that $I \trianglelefteq R$. Then we define addition and multiplication on R/I by

$$(a + I) + (b + I) = (a + b) + I \qquad \text{and} \qquad (a + I)(b + I) = ab + I.$$

Example

Suppose that $R = \mathbb{Z}[x]$ and that $I = (x^2, 2)$. What are the distinct cosets for R/I? Write out addition and multiplication tables for R/I.

Theorem

Suppose that $I \trianglelefteq R$. Then,

1. *R/I is a ring.*
2. *If R is commutative then R/I is commutative.*
3. *If R has an identity, then R/I has identity $1_R + I$.*

Definition

R/I is called the quotient ring or factor ring of R by I.

Definition

Suppose that $f : R \to S$ is a homomorphism of rings. We define the kernel of f as

$$\ker(f) = \{r \in R : f(r) = 0_S\}.$$

Theorem

Suppose that $f : R \to S$ is a homomorphism of rings. Then $\ker(f) \trianglelefteq R$.

Theorem

Suppose that $f : R \to S$ is a homomorphism of rings. Then $\ker(f) = \{0\}$ if and only if f is injective.

Theorem

Suppose that $I \trianglelefteq R$. Then the map $\pi : R \to R/I$ given by $\pi(r) = r + I$ is a surjective ring homomorphism with kernel I.

Definition

The map π is called the natural homomorphism from R to R/I.

Definition

Suppose that $f : R \to S$ is a homomorphism of rings. Then we say that S is a homomorphic image of R.
If f is an isomorphism then we say that S is an isomorphic image of R.

Theorem (First Isomorphism Theorem)

Suppose that $f : R \to S$ is a surjective homomorphism of rings with kernel K. Then $R/K \cong S$.

Example

Consider the ring $R = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$. Let $I = \{f \in R \mid f(2) = 0 = f(3)\}$. Describe R/I.

Example

Describe all homomorphic images of $\mathbb{Z}$.