MTHSC 412 Section 6.3 – The structure of R/I when I is prime or maximal

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Definition

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5. (x) is a prime ideal of $\mathbb{Z}[x]$.

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Suppose that R is a commutative ring with identity and that $P \trianglelefteq R$. Then $a + P = 0_R + P$ in R/P if and only if $a \in P$.

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3. $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$ is not a field. What is different here?

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Corollary

In a commutative ring with identity every maximal ideal is prime.