MTHSC 412 Section 6.3 – The structure of R/I when I is prime or maximal

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Definition

Suppose that R is a commutative ring and that $P \trianglelefteq R$. P is said to be prime if $P \neq R$ and whenever $bc \in P$, either $b \in P$ or $c \in P$.

Example

1. (3) is prime in $\mathbb{Z}$. (6) is not a prime ideal of $\mathbb{Z}$.
2. (0_R) is a prime ideal if R is an integral domain.
3. If F is a field and if $p(x)$ is irreducible in $F[x]$, then (p) is a prime ideal of $F[x]$ (see Thm. 4.11).
4. Let $R = \mathbb{Z}[x]$ and let $I = \{f \in R : 3|f(0)\}$. Then I is not principal. Thus $I \neq R$. I is in fact prime.
5. (x) is a prime ideal of $\mathbb{Z}[x]$.

Lemma

Suppose that R is a commutative ring with identity and that $P \trianglelefteq R$. Then $a + P = 0_R + P$ in R/P if and only if $a \in P$.

Theorem

Suppose that R is a commutative ring with identity and that $P \trianglelefteq R$. P is prime if and only if R/P is an integral domain.

Example

1. $\mathbb{Z}/(3) = \mathbb{Z}_3$ is a field.
2. $\mathbb{R}[x]/(x^2 + 1)$ is a field
3. $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$ is not a field. What is different here?

Definition

Suppose that $M \trianglelefteq R$. M is said to be maximal in R if $M \neq R$ and whenever $M \subseteq J \trianglelefteq R$ then either $J = M$ or $J = R$.

Example

1. (5) is maximal in $\mathbb{Z}$.
2. $x^2 + 1$ is maximal in $\mathbb{R}[x]$.

Theorem

Suppose that R is a commutative ring with identity and that $M \trianglelefteq R$. Then M is maximal in R if and only if R/M is a field.

Corollary

In a commutative ring with identity every maximal ideal is prime.