MTHSC 412 Section 7.1 – Definitions and Examples of Groups

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Definition

A permutation of a set T is a bijection $f : T \to T$.

Notation

Suppose that $T = \{1, 2, 3\}$ and that $f : T \to T$ is defined by $f(1) = 3$, $f(2) = 1$ and $f(3) = 2$.

We denote f by $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

Suppose that g is the map denoted by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Compute $f \circ g$.

Definition

We denote the set of all permutations from $\{1, 2, \ldots, n\}$ as S_n.

Fact

1. *If* $f, g \in S_n$ *then* $f \circ g \in S_n$.
2. $(f \circ g) \circ h = f \circ (g \circ h)$ *for all* $f, g, h \in S_3$.
3. *The identity permutation* $\begin{pmatrix} 1 & 2 & \ldots n & \\ 1 & 2 & \ldots & n \end{pmatrix}$ *is an identity with respect to the binary operation* $\circ$.
4. *Every bijection* $f \in S_n$ *has an inverse with respect to* $\circ$. *In fact the inverse of* $\begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$ *is* $\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 2 & \ldots & n \end{pmatrix}$, *where we would probably rearrange the columns of the last matrix.*

Definition

A group is a nonempty set G along with a binary operation $*$ which satisfies the following axioms.

Closure If $a, b \in G$ then $a * b \in G$.

Associativity If $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.

Identity Element There is an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$.

Inverses For each $a \in G$ there is an element $b \in G$ called the inverse of a which satisfies $a * b = b * a = e$.

A group is called abelian if it also satisfies the following axiom

Commutativity For all $a, b \in G$, $a * b = b * a$.

Definition

- A group is said to have finite order if it has a finite number of elements. In this case, the number of elements of G is denoted $|G|$ and is called the order of G.
- A group with infinitely many elements is said to be of infinite order.

Example

1. S_n is a group under composition of functions.
2. $\mathbb{Z}$ is an abelian group under addition.
3. $\mathbb{Z}_n$ is an abelian group under addition.
4. $\mathbb{Q} - \{0\}$ is an abelian group under multiplication.
5. If S is any set (even an infinite one) then the set of permutations of S is a group under composition of functions.
6. Symmetry groups

Definition

The group of rigid motions of a regular n-gon is called the Dihedral group of degree n and is denoted by D_n.

Theorem

Every ring is an abelian group under addition.

Theorem

If R is a ring with identity, then the set R^* of units of R is a group under multiplication.

Corollary

The nonzero elements of a field form an abelian group under multiplication.

Example

1. Let U_n denote the set of units in $\mathbb{Z}_n$. Write out the multiplication table for U_9.
2. $\mathbb{GL}_n(\mathbb{R}) = \{A \in \mathbb{M}_n(\mathbb{R}) : A \text{ is invertible}\}$.

Theorem

Let $(G, *)$ and $(H, \circ)$ be groups. Then $G \times H$ is a group with operation defined by $(g_1, h_1)(g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$. If G and H are abelian then so is $G \times H$. If G and H are finite then so is $G \times H$ and $|G \times H| = |G||H|$.