MTHSC 412 Section 7.4 – Isomorphisms and Homomorphisms

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Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups. A map $f : G \to H$ is said to be a homomorphism if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in G$.

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Note

Isomorphism is an equivalence relation.

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5. $S_3 \ncong \mathbb{Z}_6$.
6. Suppose that G is a group and $c \in G$. The map $f : G \to G$ given by $f(g) = c^{-1}gc$ is an automorphism of G called the inner automorphism of G induced by c.

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Suppose that G *and* H *are groups with identity elements* e_G *and* e_H *respectively. Suppose also that* $f : G \to H$ *is a homomorphism. Then,*

1. $f(e_G) = e_H$.
2. *For all* $a \in G$, $f(a^{-1}) = f(a)^{-1}$.
3. $Im(f) \leq H$.
4. *If* f *is injective, then* $G \cong Im(f)$.

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Corollary

Every finite group G of order n is isomorphic to a subgroup of S_n.