MTHSC 412 Section 7.4 – Isomorphisms and Homomorphisms

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Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups. A map $f : G \to H$ is said to be a homomorphism if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in G$.

Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups, and that $f : G \to H$ is a homomorphism. We say that f is an isomorphism if f is a bijection as well.
If an isomorphism exists from G to H or vice versa, then we say that G and H are isomorphic and write $G \cong H$.

Note

Isomorphism is an equivalence relation.

Example

1. $U_9 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$.
2. $(\mathbb{R}_{\geq 0}, +) \cong (\mathbb{R}^*, \cdot)$.
3. If $f : R \to S$ is an isomorphism of rings, then f is also an isomorphism from the group $(R, +)$ to the group $(S, +)$.
4. Define $f : \mathbb{Z} \to 3\mathbb{Z}$ by $f(z) = 3z$. Then, f is a group isomorphism. However there is no ring isomorphism between these rings.
5. $S_3 \ncong \mathbb{Z}_6$.
6. Suppose that G is a group and $c \in G$. The map $f : G \to G$ given by $f(g) = c^{-1}gc$ is an automorphism of G called the inner automorphism of G induced by c.

Theorem

Every infinite cyclic group is isomorphic to $\mathbb{Z}$*.*
Every finite cyclic group of size n *is isomorphic to* $\mathbb{Z}_n$*.*

Theorem

Suppose that G *and* H *are groups with identity elements* e_G *and* e_H *respectively. Suppose also that* $f : G \to H$ *is a homomorphism. Then,*

1. $f(e_G) = e_H$.
2. For all $a \in G$, $f(a^{-1}) = f(a)^{-1}$.
3. $Im(f) \leq H$.
4. If f is injective, then $G \cong Im(f)$.

Theorem (Cayley)

Every group G is isomorphic to a group of permutations.

Corollary

Every finite group G of order n is isomorphic to a subgroup of S_n.