Examples of Groups

Kevin James

Definition

The Dihedral group D_{2n} is the set of rigid motions on the regular n-gon. That is, D_{2n} is the the set of adjacency preserving bijections from the n-gon to itself.

Fact

D_{2n} is a group under composition of functions and $|D_{2n}| = 2n$.

Definition

- Define r to be the map that rotates the n-gon clockwise by $\frac{2\pi}{n}$ radians.
- Define s to be the map that reflects the n-gon across the diagonal passing through the vertex labelled 1.

Fact

1. *The maps $1 = r^0, r, r^2, \ldots, r^{n-1}$ are distinct and $r^n = 1$. Thus $|r| = n$*
2. $|s| = 2$.
3. *The maps $s, sr, sr^2, \ldots, sr^{n-1}$ are distinct from each other and distinct from the maps r^k.*
4. $D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$.
5. $rs = sr^{-1} (= sr^{n-1})$.
6. $r^k s = sr^{-k}$.

Note

$D_{2n} =< r, s \quad | \quad r^n = s^2 = 1; rs = sr^{-1} > .$

Remark

In general if G is a group and all elements of G can be written as words in the elements of some set S, and if all relations among $S \cup \{1\}$ may be derived from the relations in a set $\mathcal{R}$ then we write

$$G =< S \quad | \quad \mathcal{R} > .$$

Such an expression is called a presentation of G.

Definition

Let Ω denote any non-empty set, then we define S_Ω to be the set of bijections on Ω. If $\Omega = \{1, 2, \ldots, n\}$, then we denote S_Ω simply as S_n.

Fact

For any non-empty set Ω, S_Ω is a group under composition of functions.

We have convenient notation for working in S_n.

Definition

We denote by $(a_1, \ldots, a_k)$ the map that sends a_i to a_{i+1} for $1 \leq i \leq (n-1)$ and a_n to a_1 and leaves all other elements fixed. This map is called a k-cycle.

Fact

1. *Disjoint cycles commute.*
2. *Each element $\sigma \in S_n$ can be written as a product of disjoint cycles.*
3. $(a_1, \ldots, a_k) = (a_2, a_3, \ldots, a_k, a_1)$.
4. $(a_1, \ldots, a_k)^{-1} = (a_k, a_{k-1}, \ldots, a_2, a_1)$.

Definition

Suppose that F is a field.

1. $\mathbb{G}L_n(F) = \{A \in \mathrm{Mat}_{n\times n}(F) \quad | \quad \det(A) \neq 0\}$.
2. $\mathbb{S}L_n(F) = \{A \in \mathrm{Mat}_{n\times n}(F) \quad | \quad \det(A) = 1_F\}$.

Fact

Both $\mathbb{G}L_n(F)$ and $\mathbb{S}L_n(F)$ are groups under matrix multiplication.

Definition

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Remark

Q_8 is a group with multiplication defined as follows.

- $1 * a = a$ for all $a \in Q_8$.
- $(-1) * (-1) = 1$; for $a = i, j, k$, $(-1) * a = -a$ and $(-1) * (-a) = a$.
- $ij = k$; $jk = i$; $ki = j$.
- $ji = -k$; $kj = -i$; $ik = -j$.