Homomorphisms and Isomorphisms of Groups

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Definition

Suppose that $(G, *)$ and $(H, \circ)$ are groups. A function $\phi : G \to H$ is a group homomorphism if

$$\phi(g_1 * g_2) = \phi(g_1) \circ \phi(g_2), \qquad \text{for all } g_1, g_2 \in G.$$

If in addition, ϕ is a bijection, then it is called a group isomorphism and the groups G and H are said to be isomorphic written $G \cong H$.

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Example

1. $G \cong G$.
2. $(\mathbb{R}, +) \cong (\mathbb{R}^+, *)$ under the isomorphism $x \to \exp(x)$
3. If $|\Delta| = |\Omega|$, then $S_\Delta \cong S_\Omega$, **Prove this!!!**

Remark

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Fact

Suppose that G and H are groups and that $G \cong H$. Then,

1. $|G| = |H|$.
2. *G is Abelian if and only if H is Abelian.*
3. *If ϕ is an isomorphism from G to H, the for all $x \in G$,* $|x| = |\phi(x)|$.

Remark

If $G =< g_1, \ldots, g_k \quad | \quad r_1, r_2, \ldots, r_m >$ and H are groups with $h_1, \ldots, h_k \in H$ satisfying the relations r_i with g_i replaced by b_i. Then, there is a unique homomorphism $\phi : G \to H$ mapping a_i to b_i.

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Example

Suppose that $k|n$. Then there is a homomorphism $\phi : D_{2n} \to D_{2k}$.