Modules

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Definition

Let R be a ring. A left R module is a set M with two operations

Addition: $+ : M \times M \to M$, and

Scalar Multiplication: $\cdot : R \times M \to M$,

satisfying

1. $(M, +)$ is an Abelian group.
2. $(rs)m = r(sm)\ \forall r, s \in R; m \in M$.
3. $(r + s)m = rm + sm$, $\forall r, s \in R; m \in M$.
4. $r(m + n) = rm + rn\ \forall r \in R; m, n \in M$.
5. If R has 1_R, then we require that $1_R m = m$, $\forall m \in M$.

Note

1. Right R-modules can be defined similarly.
2. Not all left R-modules are right R-modules and vice versa.
3. Modules satisfying $1_R m = m$ are called <u>unital modules</u>.
4. If R is a field, then M is an R-module if and only if it is a vector space over R.

Definition

Suppose that M is an R-module. An R-submodule of M is a subgroup $N \leq M$ which is closed under the action of ring elements (-i.e. $\forall r \in R; n \in N, rn \in N$.)

Example

1. R is an R-submodule (left and right). The ideals of R are R-submodules. If R is not commutative then we may get different right and left modules.
2. $\mathbb{F}^n$ is an $\mathbb{F}$-module.
3. R^n is an R-module. The sets $S_i = \{(0_R, \ldots, 0_R, r_i, 0_r, \ldots, 0_R) \quad | \quad r_i \in R\}$ are R-submodules.
4. $\mathbb{R}$ is an $\mathbb{R}$-module, a $\mathbb{Q}$-module and a $\mathbb{Z}$-module.
5. Suppose that M is an R-module and $I \trianglelefteq R$ which annihilates M. Then, M is also a R/I-module with $(r + I)m = rm$.

Fact

Any Abelian group is a $\mathbb{Z}$-module. Its $\mathbb{Z}$-submodules are the subgroups.

Note

1. If A is an Abelian group and $x \in A$ with $|x| = n$ then $nx = 0_A$. This does not happen in vector spaces.
2. If $|A| = m$, then $mx = 0_A$, $\forall x \in A$. Thus $m\mathbb{Z}$ annihilates A. So, A is a $\mathbb{Z}/m\mathbb{Z}$-module.
3. If p is a prime and $px = 0$ for all $x \in A$, then A is a $\mathbb{Z}/p\mathbb{Z}$-module, that is a vector space over $\mathbb{Z}/p\mathbb{Z}$.
4. The Klein 4 group is a vector space over $\mathbb{Z}/2\mathbb{Z}$.

Example

Suppose that V is a vector space over the field F and $T : V \to V$ is a linear transformation. Then we can place a $F[x]$-module structure on V by defining

$$\left(\sum_{n=0}^{d_f} f_n x^n\right) \cdot v = \sum_{n=0}^{d_f} f_n T^n(v).$$

Proposition (Submodule Criterion)

Suppose $N \subseteq M$. N is a submodule of M if and only if

1. $N \neq \emptyset$, *and*
2. $x + ry \in N, \forall r \in R; x, y, N.$

Definition

Suppose that R is a commutative ring with 1_R. An R-Algebra is a ring A with 1_A with a homomorphism $f : R \to A$ mapping 1_R to 1_A and such that $f(R) \subset \text{Center}(A)$.

Note

We can put an R-module structure on an R-algebra A by defining

$$r \cdot a = f(r)a.$$

Definition

If A and B are R-algebras, an R-algebra homomorphism is a ring homomorphism $\phi : A \to B$ mapping $1_a \to 1_B$ such that $\phi(ra) = r\phi(a), \forall r \in R; a \in A$.

Example

1. A ring with 1 is a $\mathbb{Z}$ algebra.
2. If $1_A \in R \subseteq \text{Center}(A)$ then A is an R-algebra.
3. Suppose that R is a commutative ring with 1_R. Then $R[x]$ is an R-algebra.