Quotient Modules and Module Homomorphsims

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Definition

Let R be a ring and let M, N be R-modules.

1. A map $\phi : M \to N$ is an R-module homomorphism provided
   1. $\phi(x + y) = \phi(x) + \phi(y)$, $\forall x, y, \in M$, and
   2. $\phi(rx) = r\phi(x)$, $\forall r \in R; x \in M$.
2. An R-module isomorphism is an R-module homomorphism which is also bijective.
3. Suppose $\phi : M \to N$ is an R-module homomorphism.
$$\ker(\phi) = \{m \in M \quad | \quad \phi(m) = 0_N\}.$$
$$\phi(M) = \{\phi(m) \quad | \quad m \in M\}.$$
4. $\mathrm{Hom}_R(M, N) =$
$\{\phi : M \to N \quad | \quad \phi \text{ is an } R\text{-module homomorphism}\}$.

Note

For an R-module homomorphism $\phi : M \to N$,
$\ker(\phi)$ is a submodule of M and
$\phi(M)$ is a submodule of N.

Proposition

Suppose that M, N are R-modules

1. $\phi : M \to N$ *is an R-module homomorphism if and only if* $\phi(rx + xy) = r\phi(x) + s\phi(y)$, $\forall r, s \in R; x, y \in M$.
2. $Hom_R(M, N)$ *is and Abelian group under point-wise addition of functions.*
3. *If R is a commutative ring, then $Hom_R(M, N)$ is an R-module with multiplication*

$$(r\phi)(m) = r\phi(m).$$

4. *If* $\phi \in Hom_R(L, M)$ *and* $\psi \in Hom_R(M, N)$, *then* $\psi \circ \phi \in Hom_R(L, N)$.
5. $Hom_R(M, M)$ *is a ring with* 1. *When R is commutative,* $Hom_R(M, M)$ *is an R-Algebra.*

Definition

$\mathrm{Hom}_R(M, N)$ is called the endomorphism ring of M and is typically denoted $\mathrm{End}_R(M)$ or simply $\mathrm{End}\, M$.

Note

1. If R is commutative, then we can define $\iota : R \to \mathrm{End}_R(M)$ by $\iota(r) = r \cdot \mathrm{Id}_M$.
2. $\iota(R) \subseteq \mathrm{Center}(\mathrm{End}\, M)$.
3. If $1_R \in R$, then $\mathrm{End}_R(M)$ is an R-Algebra.
4. If R is a field then ι is injective and $\iota(R)$ is called the subring of scalar maps.

Proposition

Let R be a ring and let M be an R-module and $N \leq M$. The group M/N can be given an R-module structure by defining

$$r(x+N) = rx + N, \quad \forall r \in R; (x+N) \in M/N.$$

The projection map $\pi : M \to N$ is a surjective R-module homomorphism.

Definition

Let A, B be R-submodules of and R-module M. Then we define

$$A + B = \{a + b \quad | \quad a \in A, b \in B\}.$$

Fact

$A + B$ is the smallest R-submodule of M containing both A and B.

Theorem (Isomorphism Theorems)

1. *Suppose that* $\phi : M \to N \in Hom_R(M,N)$. *Then,* $\ker(\phi) \leq M$ *and* $M/\ker(\phi) \cong \phi(M)$.
2. *Suppose that* $A, B \leq M$. *Then,*
$$(A+B)/B \cong A/(A \cap B).$$
3. *Suppose that* $A, B \leq M$ *and* $A \subseteq B$. *Then,*
$$\frac{M/A}{M/B} \cong M/B.$$
4. *Suppose that* $N \leq M$. *Then,*
$$\{N \leq K \leq M\} \xleftrightarrow{1-1} \{L \leq M/N\}$$
under the map $K \mapsto K/N$.
This correspondence commutes with sums and intersections.