Tensor Products of Modules

Kevin James

Definition

Suppose that S and N are R modules. Let $A = S \times N$;
$G = F_{\mathbb{Z}}(A) = \left\{ \sum_{i=1}^{k} (s_i, n_i) \right\}$;

$$H = \left\langle \begin{matrix} [(s_1 + s_2, n) - (s_1, n) - (s_2, n)], \\ [(s, n_1 + n_2) - (s, n_1) - (s, n_2)], \\ [(sr, n) - (s, rn)] \end{matrix} \;\middle|\; \begin{matrix} s, s_1, s_2 \in S; \\ n, n_1, n_2 \in N; \\ r \in R \end{matrix} \right\rangle.$$

Then,

$$S \otimes_R N = G/H.$$

Note

1. We denote the coset of $S \otimes_R N$ containing (s, n) as $s \otimes n$.
2. By definition, we have

$$\begin{aligned} (s_1 + s_2) \otimes n &= s_1 \otimes n + s_2 \otimes n \\ s \otimes (n_1 + n_2) &= s \otimes n_1 + s \otimes n_2 \\ sr \otimes n &= s \otimes rn \end{aligned}$$

3. Elements of $S \otimes_R N$ are called tensors and can be written as

$$\sum_{i=1}^{k} s_i \otimes n_i.$$

Fact

We define $\cdot : S \times S \otimes_R N \to S \otimes_R N$ *by* $s \cdot \sum_{i=1}^{k} s_i \otimes n_i = \sum_{i=1}^{k} ss_i \otimes n_i$. *Under this operation* $S \otimes_R N$ *is an* S*-module.*