Algebraic Extensions

Kevin James

Definition

Suppose that $K \supseteq F$ is an extension of fields. Then $\alpha \in K$ is algebraic over F if $\exists f(x) \in F[x]$ such that $f(\alpha) = 0$. Otherwise, α is transcendental over F.

Proposition

Let α be algebraic over F. Then there is a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m_{\alpha,F}(\alpha) = 0$. Further, if $f \in F[x]$ has $f(\alpha) = 0$, then $m_{\alpha,F}(x) | f(x)$ in $F[x]$.

Corollary

Suppose that L/F is an extension of fields and α is algebraic over F and L. Then $m_{\alpha,L}(x) | m_{\alpha,F}(x)$ in $L[x]$.

Definition

The polynomial $m_{\alpha,F}(x)$ is the minimum polynomial of α over F, and we put

$$\deg(\alpha) = \deg(m_{\alpha,F}(x).$$

Proposition

Suppose that α is algebraic over F. Then,

$$F(\alpha) \cong F[x]/m_{\alpha,F}(x).$$

Also, $[F(\alpha) : F] = \deg(m_{\alpha,F}(x)) = \deg(\alpha).$

Proposition

We note that α is algebraic over F if and only if $[F(\alpha) : F] < \infty$. More precisely, we have the following. If $\alpha \in K$ and $[K : F] = n$, then $\deg(\alpha) \leq n$. If $\deg(\alpha) \leq n$ then $[F(\alpha) : F] \leq n$.

Corollary

If $[K : F] < \infty$, then K is algebraic over F (-i.e. every element of K is algebraic over F).

Theorem

Suppose that $F \subseteq K \subseteq L$ are fields. Then $[L : F] = [L : K][K : F]$.

Corollary

Suppose that L/F is a finite extension and that $F \subseteq K \subseteq L$. Then $[K:F] | [L:F]$.

Corollary

If $[L:F] = p$ is prime, then thre are no intermediate extensions.

Definition

We say that the extension K/F is finitely generated if there exists $\alpha_1, \dots, \alpha_k \in K$ such that $K = F(\alpha_1, \dots, \alpha_k)$.

Note

Note that F/K finitely generated does **NOT** imply that F/K is finite dimensional.

Lemma

$F(\alpha, \beta) = F(\alpha)(\beta)$.

Theorem

The extension K/F is finite if and only if $K = F(\alpha_1, \alpha_2, \ldots, \alpha_k)$ where each α_i is algebraic over F. More precisely if $\alpha_1, \ldots, \alpha_k$ are algebraic over F, then $[F(\alpha_1, \alpha_2, \ldots, \alpha_k) : F] \leq \prod_{i=i}^{k} \deg(\alpha_i)$.

Corollary

Suppose that α and β are algebraic over a field F. Then so are $\alpha \pm \beta$, $\alpha\beta$, α/β ($\beta \neq 0$), α^{-1} ($\alpha \neq 0$).

Corollary

Let L/F be an arbitrary extension of fields. Then $K = \{\alpha \in L \mid \alpha \text{ is algebraic over } F\}$ is a field with $F \subseteq K \subseteq L$.

Theorem

If K is algebraic over F and L is algebraic over K, then L is algebraic over F.

Definition

Let K be a field containing subfields K_1 and K_2. We define the composite of K_1 and K_2 denoted K_1K_2 to be the smallest subfield of K containing both K_1 and K_2.

Proposition

Suppose that K_1 and K_2 are finite extensions of F contained in a field K. Then,

$$[K_1K_2 : F] \leq [K_1 : F][K_2 : F],$$

with equality if and only if and F-basis for one of K_1 or K_2 remains linearly independent over the other. If $\alpha_1, \dots, \alpha_n\}$ and $\beta_1, \dots, \beta_m\}$ are bases for K_1 and K_2 respectively as vector spaces over F, then $\{\alpha_i\beta_j\}_{1\leq i\leq n, 1\leq j\leq m}$ spans K_1K_2.

Corollary

Suppose that $[K_1 : F] = n$; $[K_2 : F] = m$ *where* $(m, n) = 1$. *Then* $[K_1K_2 : F] = mn$.