Splitting Fields and Algebraic Closures

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Definition

The extension K/F is called the splitting field for the polynomial $f(x) \in F[x]$ if $f(x)$ factors completely into linear factors in $K[x]$ and $f(x)$ does not factor into linear factors over $L[x]$ for any $K \supset L \supseteq F$.

Theorem

For any field F and any $f(x) \in F[x]$, there exists an extension K/F which is a splitting field for $f(x)$.

Definition

If K is an algebraic extension of F which is the splitting field over F for a collection of polynomials $f(x) \in F[x]$ then K is called a normal extension of F.

Proposition

A splitting field of a polynomial of degree n over F is of degree at most $n!$ over F.

Theorem

Let $\phi : F \to F'$ be an isomorphism of fields. let $f(x) \in F[x]$ and let $f'(x) = \phi(f(x)) \in F'[x]$. Let E/F be a splitting field of f and let E'/F' be a splitting field of f'. The isomorphism ϕ extends to an isomorphism $\sigma : E \to E'$.

Corollary (Uniqueness of Splitting Fields)

Any two splitting fields for a polynomial $f(x) \in F[x]$ over F are isomorphic.

Definition

The field $\bar{F}$ is called an algebraic closure of F if $\bar{F}$ is algebraic over F and if every polynomial $f(x) \in F[x]$ splits completely over $\bar{F}$

Definition

A field K is said to be algebraically closed if all polynomials $f(x) \in K[x]$ have a root in K.

Proposition

Let $\bar{F}$ be an algebraic closure of F. Then $\bar{F}$ is algebraically closed.

Proposition

For any field F there exists an algebraically closed field K containing F.

Proposition

Let K be an algebraically closed field and let F be a subfield of K. Then the collection of elements $\bar{F}$ of K which are algebraic over F is an algebraic closure of F. An algebraic closure of F is unique up to isomorphism.

Theorem (Fundamental Theorem of Algebra)

The field $\mathbb{C}$ is algebraically closed.

Corollary

The field $\mathbb{C}$ contains an algebraic closure for any of it s subfields. In particular, $\bar{\mathbb{Q}}$, the collection of complex numbers algebraic over $\mathbb{Q}$, is an algebraic closure of $\mathbb{Q}$.