Splitting Fields and Algebraic Closures

Kevin James

Definition

The extension K/F is called the splitting field for the polynomial $f(x) \in F[x]$ if $f(x)$ factors completely into linear factors in $K[x]$ and $f(x)$ does not factor into linear factors over $L[x]$ for any $K \supset L \supseteq F$.

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Theorem

For any field F and any $f(x) \in F[x]$, there exists an extension K/F which is a splitting field for $f(x)$.

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If K is an algebraic extension of F which is the splitting field over F for a collection of polynomials $f(x) \in F[x]$ then K is called a normal extension of F.

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Let $\phi : F \to F'$ be an isomorphism of fields. let $f(x) \in F[x]$ and let $f'(x) = \phi(f(x)) \in F'[x]$. Let E/F be a splitting field of f and let E'/F' be a splitting field of f'. The isomorphism ϕ extends to an isomorphism $\sigma : E \to E'$.

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Corollary (Uniqueness of Splitting Fields)

Any two splitting fields for a polynomial $f(x) \in F[x]$ over F are isomorphic.

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For any field F there exists an algebraically closed field K containing F.

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The field $\mathbb{C}$ is algebraically closed.

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Theorem (Fundamental Theorem of Algebra)

The field $\mathbb{C}$ is algebraically closed.

Corollary

The field $\mathbb{C}$ contains an algebraic closure for any of it s subfields. In particular, $\bar{\mathbb{Q}}$, the collection of complex numbers algebraic over $\mathbb{Q}$, is an algebraic closure of $\mathbb{Q}$.