Centralizers, Normalizers, Stabilizers and Kernels

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Definition

Suppose that G is a group and that $\emptyset \neq A \subseteq G$. We define the <u>centralizer</u> of A in G as

$$C_G(A) = \{g \in G \quad | \quad gag^{-1} = a, \forall a \in A\}.$$

Note

$C_G(A)$ is the set of elements of G which commute with each element of A.

Proposition

Given a group G and $\emptyset \neq A \subseteq G$, $C_G(A) \leq G$.

Definition

We define the center of a group G as $Z(G) = C_G(G)$ and note that it is the set of elements of G which commute with all other elements.

Definition

Given a group G and $\emptyset \neq A \subseteq G$, we define for $g \in G$, $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. Further, we define the normalizer of A in G to be $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$.

Note

Given a group G and $\emptyset \neq A \subseteq G$ note that $Z(G) \leq C_G(A) \leq N_G(A) \leq G$.

Example

Take $A = \{1, r, r^2, r^3\} \leq D_8$. Prove that

1. $C_{D_8}(A) = A$,
2. $N_{D_8}(A) = D_8$, and
3. $Z(D_8) = \{1, r^2\}$.

Definition

Suppose that S is a nonempty set and that G is a group acting on S.

1. For $s \in S$, we define the <u>stabilizer</u> of s to be $G_s = \{g \in G \quad | \quad g \cdot s = s\}$, and
2. we define the <u>kernel</u> of the action to be $\{g \in G \quad | \quad g \cdot s = s, \forall s \in S\}$.

Fact

Suppose that S is a nonempty set and that G is a group acting on S. For any $s \in S$, $G_s \leq G$. Also the kernel of the action is a subgroup of G.

Example

Let $G = D_8$ and let $S = \{1, 2, 3, 4\}$. For any $s \in S$, $G_s = \{1, t\}$ where t denotes the reflection about the diagonal passing through s.

Remark

1. Let $S = 2^G$ (the power set of G) and let G act on S by conjugation (-i.e. $g \cdot s = gsg^{-1}$. If $A = s \in S$ then $N_G(A) = G_s$.
2. Let $N_G(A)$ act on $S = A$ by conjugation. Then $C_G(A)$ is the kernel of the action.
3. Let G act on $S = G$ by conjugation, then the kernel of the action is $Z(G)$.