Quotient Groups and Homomorphisms: Definitions and Examples

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Definition

If $\phi : G \to H$ is a homomorphism of groups, the kernel of ϕ is the set

$$\ker(\phi)\{g \in G \quad | \quad \phi(g) = 1_H\}.$$

Proposition

Suppose that $\phi : G \to H$ is a group homomorphism.

1. $\phi(1_G) = 1_H$.
2. $\phi(g^{-1} = (\phi(g))^{-1}$, $\forall g \in G$.
3. $\phi(g^n) = \phi(g)^n$, $\forall g \in G$.
4. $\ker(\phi) \leq G$.
5. $im(\phi) \leq H$.

Definition

Suppose that $\phi : G \to H$ is a homomorphism of groups. For $h \in H$, we define the fiber above h to be $X_h = \phi^{-1}(h)$.

Proposition

Suppose that $\phi : G \to H$ *is a homomorphism of groups and let* $K = \ker(\phi)$*, we denote by* G/K *the set of all fibers. The set* G/K *is a group with operation defined by* $X_a X_b = X_{ab}$*. This group is called the quotient group of* G *by* K

Proposition

Suppose that $\phi : G \to H$ is a homomorphism of groups and let $K = \ker(\phi)$. Let $X \in G/K$. Then,

1. *For any $u \in X$, $X = \{uk \mid k \in K\} := uK$.*
2. *For any $u \in X$, $X = \{ku \mid k \in K\} := Ku$.*

Definition

For any $N \leq G$ and $g \in G$ let

$$gN = \{gn \mid n \in N\} \qquad Ng = \{ng \mid n \in N\}.$$

called respectively a left coset of N and a right coset of N.

Theorem

Let G be a group and let $K \leq G$ be the kernel of some homomorphism from G to some other group. The set of left cosets of K with operation defined by $(uK) \cdot (wK) = uwK$ forms a group G/K. The same is true if we replace "left coset" by "right coset."

Proposition

Let $N \leq G$. The set of left cosets of N in G form a partition of G. Furthermore, for all $u, w \in G$, $uN = wN$ if and only if $w^{-1}u \in N$.

Proposition

Let $N \leq G$*.*

1. *The operation defined on the set of left cosets of* N *by* $uN \cdot wN = uwN$ *is well defined if and only if* $gNg^{-1} = N$*,* $\forall g \in G$*.*
2. *In the case that the above operation is well defined, it makes* G/N *the set of left cosets of* N *into a group.*

Definition

Suppose that $N \leq G$. We say that g normalizes N if $gNg^{-1} = N$. The subgroup N is said to be normal if every element of G normalizes N. If N is a normal subgroup of G, we write $N \trianglelefteq G$.

Theorem

Let $N \leq G$. The following are equivalent.

1. $N \trianglelefteq G$.
2. $N_G(N) = G$.
3. $gN = Ng$, $\forall g \in G$.
4. *The operation* $(uN) \cdot (wN) = uwN$ *is a well defined and makes* G/N *into a group.*
5. $gNg^{-1} \subseteq N$, $\forall g \in G$.

Proposition

Suppose that $N \leq G$. *Then,* $N \trianglelefteq G$ *if and only if* $N = \ker(\phi)$ *for some homomorphism* ϕ.

Definition

Suppose that $N \trianglelefteq G$. Then the homomorphism $\pi : G \to G/N$ given by $\pi(g) = gN$ (Check that this is indeed a homomorphism.) is called the natural projection of G onto G/N.

Note

The natural projection of G onto G/N gives a one to one correspondence between the subgroups of G/N and the subgroups H of G satisfying $N \leq H \leq G$.