More on Cosets and Lagrange's Theorem

Kevin James

Theorem (Lagrange's Theorem)

Suppose that G is a group and that $H \leq G$. Then $|H|$ divides $|G|$. In fact, if we denote the number of cosets of H in G by $[G : H]$ (also called the index of H in G), then we have $|G| = [G : H]|H|$.

Corollary

Suppose that G is a group and that $x \in G$. Then $|x|$ divides $|G|$. In particular, $x^{|G|} = 1$ for all $x \in G$.

Corollary

If G is a group of prime order then G is cyclic.

Note

Note that if G is a group and $H \leq G$ with $|H| = |G|/2$, then $H \trianglelefteq G$.

Definition

Suppose that $H, K \leq G$. Define

$$HK = \{hk \quad | \quad h \in H; k \in K\}.$$

Proposition

Suppose that $H, K \leq G$ where G is a finite group. Then,

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Proposition

Suppose that $H, K \leq G$. The set HK is a subgroup of G if and only if $HK = KH$.

Corollary

If $H, K \leq G$ and $H \leq N_G(K$, then $HK \leq G$. In particular, if $K \trianglelefteq G$, then $HK \leq G$ for all $H \leq G$.